JHEP HYPER VERSION

Received: *June 15, 2000*, Accepted: *Jun 19, 2000*

Holomorphy, rescaling anomalies and exact β functions in supersymmetric gauge theories

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Abstract: There have long been known "exact" β functions for the gauge coupling in $N = 1$ supersymmetric gauge theories, the so-called Novikov-Shifman-Vainshtein-Zakharov (NSVZ) β functions. Shifman and Vainshtein further related these β functions to the exact 1-loop running of the gauge coupling in a "wilsonian" action. All these results, however, remain somewhat mysterious. We attempt to clarify these issues by presenting new perspectives on the NSVZ β function. Our interpretation of the results is somewhat different than the one given by Shifman and Vainshtein, having nothing to do with the distinction between "wilsonian" and "1PI" effective actions. Throughout we work in the context of the Wilsonian Renormalization Group; namely, as the cutoff of the theory is changed from M to M', we seek to determine the appropriate changes in the bare couplings needed in order to keep the low energy physics fixed. The entire analysis is therefore free of infrared subtleties. When the bare lagrangian given at the cutoff is manifestly holomorphic in the gauge coupling, we show that the required change in the holomorphic gauge coupling is exhausted at 1-loop to all orders of perturbation theory, and even non-perturbatively in some cases. On the other hand, when the bare lagrangian at the cutoff has canonically normalized kinetic terms, we find that the required change in the gauge coupling is given by the NSVZ β function. The higher order contributions in the NSVZ β function are due to anomalous jacobians under the rescaling of the fields done in passing from holomorphic to canonical normalization. We also give prescriptions for regularizing certain $N = 1$ theories with an ultraviolet cutoff M preserving manifest holomorphy, starting from finite $N = 4$ and $N = 2$ theories. It is then at least in principle possible to check the validity of the exact β function by higher order calculations in these theories.

Keywords: Renormalization Regularization and Renormalons, Anomalies in Field and String Theories, Supersymmetric Effective Theo ries.

JHEP06(2000)030

Contents

1. Introduction

In recent years, enormous progress has been made in understanding the non-perturbative dynamics of supersymmetric gauge theories [1]. Many theories have been solved "exactly", setting the stage for applications of strong supersymmetric gauge dynamics in building realistic models of particle physics.

However, the connections between the exact results and those obtained in perturbation theory are still not entirely clear. One famous example of a confusion in this regard is the anomaly puzzle. In supersymmetric theories, the $\mathrm{U}(1)_R$ current is in the same multiplet as the trace of the energy-momentum tensor [2], and hence the chiral anomaly and the trace anomaly are related [3, 4]. The chiral anomaly is exhausted at 1-loop [5], however, implying that the trace anomaly is exhausted also at 1-loop. Since the trace anomaly determines the gauge coupling β function $\beta(g)$, this seems to imply that this $\beta(g)$ should be also exhausted at 1-loop. However, explicit perturbative calculations find higher order corrections to $\beta(g)$ [6].

Shifman and Vainshtein [7] presented a solution to this puzzle, by distinguishing between the "wilsonian" gauge coupling constant g_W and the "1PI" or "physical" coupling g. In their interpretation, g_W appears in the wilsonian effective action and only runs at 1-loop, whereas g appears in the 1PI effective action and receives higher order corrections. Moreover, they presented a remarkable formula relating the two types of gauge coupling from which they obtained, to all orders in perturbation theory, the exact $\beta(g)$ for $N = 1$ supersymmetric gauge theories with matter fields ϕ_i. The same β function was first derived via different arguments by Novikov, Shifman, Vainshtein and Zakharov (NSVZ) [8]:

$$\beta(g) = -\frac{1}{16\pi^2}\frac{3t_2(A) - \sum_i t_2(i)(1-\gamma_i)}{1 - t_2(A)g^2/8\pi^2}\,. \tag{1.1}$$

Here $t_2(A), t_2(i)$ are the Dynkin indices for the adjoint and ϕ_i representations (e.g. $t_2(\mathbf{N}) = 1/2$ and $t_2(A) = N$ in SU(N)), and γ_i is the anomalous dimension of ϕ_i. Explicit perturbative calculations verify the NSVZ β function up to two-loop order. This is clearly a significant achievement. Beyond two loops, however, the β function coefficients are scheme dependent, and it is not clear in what scheme the NSVZ β function is supposed to be exact.[1] This is one aspect of a general confusion (which at least we have) surrounding the arguments leading to the NSVZ β function.

The purpose of this paper is to attempt to eliminate these confusions by giving independent derivations of the NSVZ β function. Our interpretation of the results, however, is somewhat different. We do not use the 1PI effective action to define the gauge coupling constant. Instead, we work throughout in the context of the Wilsonian Renormalization Group (WRG), which we briefly review here.[2] Any field theory is defined with some cutoff M, and bare couplings λ_0^i. If we wish to change the cutoff from M to M' while keeping the low energy physics fixed (this step is often referred to as "integrating out modes between M and M'"), we need to change the bare couplings $\lambda_i^0 \to \lambda_i^{0\prime}$. The way in which the λ_i^0 must change with the cutoff M keeping the low energy physics fixed is encoded in a Wilsonian Renormalization Group Equation (WRGE) for the λ_i^0, $(Md/dM)\lambda_i^0 = \beta_i(\lambda^0)$. All of the usual results of renormalization-group analysis can be derived along these lines (see [11] for some examples). The virtue of this approach is the freedom from infrared subtleties. All the modes beneath M' have yet to be integrated over, so none of the calculations involve infrared divergences. Since the infrared effects are sensitive to the

[1] One can relate $\overline{\mathrm{DR}}$ scheme and the NSVZ scheme order by order in perturbation theory [9].

[2] For a general discussion of wilsonian renormalization program in continuum field theories, see [10]. See also [11] for a more complete discussion of the WRG invariance of exact results in SUSY gauge theories.

detailed dynamics of different models, it is difficult to make exact and non-trivial statements on the evolution of the coupling constant if the calculation involves infrared effects. By separating the infrared physics from the discussion, we will be able to make concrete statements on the ultraviolet structure of supersymmetric gauge theories with confidence. Having understood the ultraviolet properties, the interesting physics lies in infrared non-perturbative dynamics, which as we know can vary drastically depending on the particular supersymmetric gauge theory under consideration.

In supersymmetric theories, we have two natural choices for the form of the lagrangian defined with cutoff M. The first is manifestly holomorphic in the combination $1/g_h^2 = 1/g^2 + i\theta/8\pi^2$. The second uses canonically normalized kinetic terms for all fields; in this case the gauge coupling is called the canonical gauge coupling g_c. We will show that, in changing the cutoff from M to M', the change in $1/g_h^2$ needed to keep the low energy physics fixed is exhausted at 1-loop, but that g_c must be changed according to the NSVZ β function. Furthermore, some special theories can be explicitly regulated in a way that preserves holomorphy. The validity of the exact β function can then at least in principle be checked by perturbative calculations in these theories.

The outline for the paper is as follows. In section 2, we consider pure SUSY Yang-Mills theories, and show that the running of the holomorphic gauge coupling $1/g_h^2$ is exhausted at 1-loop. However, in the rescaling of the vector multiplet needed to go to canonical normalization, we encounter an anomalous jacobian. Correctly accounting for this anomalous jacobian gives the relation between g_c and g_h given by Shifman and Vainshtein, and hence the exact NSVZ β-function. In section 3, we address the anomaly puzzle in our framework. The resolution is very simple. The anomaly under dilations (trace anomaly) is in the same multiplet as the $\mathrm{U}(1)_R$ anomaly and is one-loop exact. However, because of the anomaly, the vector multiplet does not have canonical kinetic terms after the dilation. If we wish to work with canonical kinetic terms for the vector multiplet, a further change in normalization (rescaling) of the vector multiplet must be done, which is itself anomalous. Therefore, the anomaly from this "modified" dilation (naive dilation + rescaling of vector multiplet) is *not* in the same multiplet as the $\mathrm{U}(1)_R$ anomaly, and receives contributions beyond 1-loop according to the NSVZ β function. In section 4, we extend the discussion on β functions to the case with matter fields. In section 5 we consider $N = 2$ theories, and use our results to explain the finiteness of these theories beyond 1-loop. In section 6, we give explicit prescriptions for regularizing some $N = 1$ theories with a cutoff M, starting from finite $N = 4$ and $N = 2$ theories. We explicitly define the couplings $g_h(M), g_c(M)$, and show that the Shifman-Vainshtein relation holds between them. We draw our conclusions in section 7, while two appendices A and B contain discussions and explicit computations of all the required anomalous jacobians.

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2. Pure $N = 1$ SUSY Yang-Mills

At a certain cutoff scale M, the lagrangian for pure $N = 1$ supersymmetric Yang-Mills (SUSY YM) can be given in two different ways. With the vector multiplet $V_h = V_h^a T^a$, we can write it in a way that is manifestly holomorphic in the gauge coupling:

$$\mathcal{L}_h^M(V_h) = \frac{1}{16}\int d^2\theta \frac{1}{g_h^2} W^a(V_h) W^a(V_h) + \text{h.c.}\,, \tag{2.1}$$

where $W_\alpha^a(V)T^a = \frac{-1}{4}\bar{D}^2 e^{-2V} D_\alpha e^{2V}$,[3] and

$$\frac{1}{g_h^2} = \frac{1}{g^2} + i\frac{\theta}{8\pi^2}\,. \tag{2.2}$$

On the other hand, we can work with canonical normalization for the gauge kinetic terms. In this case, the lagrangian is written as

$$\mathcal{L}_c^M(V_c) = \frac{1}{16}\int d^2\theta \left(\frac{1}{g_c^2} + i\frac{\theta}{8\pi^2}\right) W^a(g_c V_c) W^a(g_c V_c) + \text{h.c.} \tag{2.3}$$

Note that since $g_c V_c$ is a real superfield, g_c must be real, and the lagrangian is *not* holomorphic in the combination $(1/g_c^2 + i\theta/8\pi^2)$.

Suppose we now change the cutoff from M to M'; how must the couplings be changed to keep the low energy physics fixed? The answer is particularly simple in the case of the holomorphic coupling. For the holomorphic coupling at the cutoff M', $1/g_h'^2$, let us write

$$\frac{8\pi^2}{g_h'^2} = \frac{8\pi^2}{g_h^2} + f\left(\frac{8\pi^2}{g_h^2}, \ln\frac{M}{M'}\right). \tag{2.4}$$

The function $f(8\pi^2/g_h^2, t)$ must be holomorphic in $1/g_h^2$, continuous in t, and must satisfy $f(8\pi^2/g_h^2, 0) = 0$. Since a 2π shift in θ has no effect, we must have $f(8\pi^2/g_h^2 + 2\pi i, t) = f(8\pi^2/g_h^2, t) + 2\pi n(t) i$, where $n(t)$ is an integer. However, since we know $n(0) = 0$, by continuity in t, $n(t) = 0$. Therefore $f(8\pi^2/g_h^2 + 2\pi i, t) = f(8\pi^2/g_h^2, t)$. These observations can be cast in terms of the wilsonian β function for the holomorphic gauge coupling: we must have

$$\frac{d}{dt}\left(\frac{8\pi^2}{g_h^2}\right) = \beta\left(\frac{8\pi^2}{g_h^2}\right); \qquad \beta\left(\frac{8\pi^2}{g_h^2} + 2\pi i\right) = \beta\left(\frac{8\pi^2}{g_h^2}\right). \tag{2.5}$$

Since the β function is periodic, it can be Fourier expanded

$$\beta\left(\frac{8\pi^2}{g_h^2}\right) = \sum_{n\geq 0} a_n e^{-n8\pi^2/g_h^2}\,, \tag{2.6}$$

[3] Our normalization of the vector multiplet differs from that of Wess and Bagger [12] by a factor of two, and we need to rescale it as $V_h = g_c V_c$ to go to canonical normalization.

where the sum is restricted to $n \geq 0$ so that the theory makes sense in weak coupling. The term with $n = 0$ is a constant a_0 ($= b_0$ is the one-loop β-function coefficient), and corresponds to the 1-loop law for the running of $1/g_h^2$. The terms with $n \geq 1$ can never arise in perturbation theory.

In fact, for pure SUSY YM, a stronger argument shows that the terms with $n \geq 1$ can not arise at all. Since the theory has an anomalous $\mathrm{U}(1)_R$ symmetry, if $1/g_h^2(t)$ is a solution to the WRGE, $1/g_h^2 + i\phi$ should also be a solution. This implies that $\beta(8\pi^2/g^2 + i(\theta + \phi)) = \beta(8\pi^2/g^2 + i\theta)$, and hence that $\beta(8\pi^2/g_h^2)$ is independent of $\mathrm{Im}(8\pi^2/g_h^2)$. However, any analytic function $f(z)$ which is independent of $\mathrm{Im}(z)$ is a constant. Thus, the holomorphic $1/g_h^2$ runs *exactly* at 1-loop for pure SUSY YM, even including non-perturbative effects.

It is important to note that this result does not hold for other definitions of β functions. For instance, consider a pure SU(2) $N = 2$ theory; this theory also has an anomalous $\mathrm{U}(1)_R$ symmetry, so the argument given above implies that the running of $1/g_h^2$ is exhausted at 1-loop. When the ($N = 1$) adjoint chiral superfield acquires a vev $\langle\phi\rangle = v\sigma_3$ breaking $\mathrm{SU}(2) \to \mathrm{U}(1)$, Seiberg [14] found that the effective value of the holomorphic gauge coupling of the unbroken U(1) is given by

$$\frac{1}{g_{\text{eff}}^2(v)} = \frac{1}{g_h^2(M)} - \frac{b_0}{16\pi^2} \ln \frac{v^2}{M^2} + c \left(\frac{M e^{8\pi^2/b_0 g_h^2(M)}}{v} \right)^4 + \mathcal{O}\left(e^{-16\pi^2/g_h^2(M)} \right), \quad (2.7)$$

where $b_0 = -4$ is the coefficient of the 1-loop β function, and the constant c as well as the higher order corrections have been determined by Seiberg and Witten [15]. If the β function is defined by $(vd/dv)\, g_{\text{eff}}(v) = \beta_{\text{eff}}(g_{\text{eff}})$, β_{eff} contains both 1-loop and non-perturbative corrections. On the other hand, suppose we lower the cutoff from M to M'; how should $1/g_h^2(M)$ change to keep low energy physics ($1/g_{\text{eff}}^2(v)$) fixed? It is clear that the required change in $1/g_h^2(M)$ is exhausted at 1-loop, i.e. $1/g_h^2(M') = 1/g_h^2(M) - (b_0/8\pi^2) \ln M'/M$, and so the wilsonian β function for the holomorphic gauge coupling is indeed exhausted at 1-loop in this example.

We now wish to determine the wilsonian β function for the canonical gauge coupling g_c. If we change the cutoff from M to M', how must $1/g_c^2$ be changed to keep the low energy physics fixed? At first sight, there seems to be no difficulty in going from the holomorphic to canonical normalizations: simply making the change of variable $V_h = g_c V_c$, the lagrangian seems to have canonical normalization for the vector multiplet with $g_c = g_h$. However, this is not correct, as there is an anomalous jacobian in passing from V_h to V_c; $\mathcal{D}(g_c V_c) \neq \mathcal{D}V_c$.[4]

In appendix A, we explicitly compute this jacobian, and at the end of section 5, we derive it indirectly based on the known finiteness of $N = 2$ theories beyond 1-loop.

[4]The measure $\mathcal{D}V$ is for the entire gauge sector of the theory, including the ghosts.

The two methods yield the same result:

$$\mathcal{D}(g_c V_c) = \mathcal{D}(V_c)\exp\left(\frac{1}{16}\int d^4y\int d^2\theta\frac{2t_2(A)}{8\pi^2}\ln g_c W^a(g_cV_c)W^a(g_cV_c) + \right.$$
$$\left. +\text{h.c.} + \mathcal{O}\left(\frac{1}{M^4}\right)\right), \tag{2.8}$$

where the F terms given above are exact, and $\mathcal{O}(1/M^4)$ refers to higher dimension D terms suppressed by powers of $1/M$ (the lowest dimension operator is of the form $\int d^4\theta WW\bar{W}\bar{W}/M^4$).

In a non-supersymmetric theory, it is not permissible to simply throw away higher dimension operators suppressed by powers of the cutoff. We can form relevant operators by closing the legs of higher dimension operators, and power divergences in the loops can negate the cutoff suppression of the higher dimension operators [10]. As far as physics at energy scales $E \ll M$ is concerned, however, the lagrangian with the higher dimension operators included yields the same Green's functions[5] as a different lagrangian with all higher dimension operators set to zero, but only after appropriately modifying the coefficients of the relevant operators. In our case, an important simplification occurs: the usual supersymmetric non-renormalization theorem [16] makes it impossible for the higher dimension D terms to ever produce an F term such as WW, and therefore no modification of the coefficient of WW is needed upon dropping the higher dimension D terms. Note also that since any possible contribution of the higher dimension D terms is coming from ultraviolet divergences which need to negate the cutoff suppressions, there is no worry about any subtle *infrared* singular D terms (such as $\int d^4\theta W\frac{-D^2}{4\Box}W$) being generated. This is welcome, since the non-renormalization theorem does not forbid the generation of these operators [8, 7, 17], but they are equivalent to $\int d^2\theta WW$: $\int d^4\theta W\frac{-D^2}{4\Box}W = \int d^2\theta\frac{\bar{D}^2}{4}W\frac{D^2}{\Box}W = \int d^2\theta WW$ up to surface terms since $\bar{D}^2D^2W = 16\Box W$.

With the jacobian (2.8), it is easy to derive the relationship between the holomorphic and canonical gauge couplings (the Shifman-Vainshtein formula [7]). At a fixed cutoff M, we have[6]

$$\begin{aligned} Z &= \int \mathcal{D}V_h \exp\left(-\frac{1}{16}\int d^4y\int d^2\theta\frac{1}{g_h^2}W^a(V_h)W^a(V_h) + \text{h.c.}\right) \\ &= \int \mathcal{D}(g_cV_c)\exp\left(-\frac{1}{16}\int d^4y\int d^2\theta\frac{1}{g_h^2}W^a(g_cV_c)W^a(g_cV_c) + \text{h.c.}\right) \\ &= \int \mathcal{D}V_c \exp\left(-\frac{1}{16}\int d^4y\int d^2\theta\left(\frac{1}{g_h^2} - \frac{2t_2(A)}{8\pi^2}\ln g_c\right)W^a(g_cV)W^a(g_cV) + \text{h.c.}\right). \end{aligned} \tag{2.9}$$

In order to have canonical normalization for the vector multiplet, we must have

$$\frac{1}{g_c^2} = \text{Re}\left(\frac{1}{g_h^2}\right) - \frac{2t_2(A)}{8\pi^2}\ln g_c\,, \tag{2.10}$$

[5] Up to corrections suppressed by powers in (E/M).

[6] For compactness, we do not write the gauge-fixing terms in the path integrals which follow.

which is the Shifman-Vainshtein formula. Since the difference between $1/g_h^2$ and $1/g_h'^2$ is exhausted at 1-loop, we have

$$\left(\frac{1}{g_c'^2}+\frac{2t_2(A)}{8\pi^2}\ln g_c'\right)=\left(\frac{1}{g_c^2}+\frac{2t_2(A)}{8\pi^2}\ln g_c\right)-\frac{3t_2(A)}{8\pi^2}\ln\frac{M}{M'}\,. \tag{2.11}$$

The exact NSVZ β function [8] for pure SUSY YM then follows trivially

$$M\frac{d}{dM}g_c=\beta(g_c)=-\frac{3t_2(A)}{16\pi^2}\frac{g_c^3}{1-\frac{t_2(A)}{8\pi^2}g_c^2}\,. \tag{2.12}$$

It is noteworthy that the above derivation of the exact β-function has no reference to 1PI effective actions or infrared effects. Indeed, the argument used here is exactly analogous to a familiar argument on the chiral anomaly: the QCD lagrangian with a complex mass parameter

$$\mathcal{L}=-\frac{1}{2}\operatorname{Tr}F^{\mu\nu}F_{\mu\nu}+\bar{q}i\not{D}q+(me^{i\phi}\bar{q}_Rq_L+\text{h.c.}) \tag{2.13}$$

can be brought to a lagrangian with a real mass

$$\mathcal{L}=-\frac{1}{2}\operatorname{Tr}F^{\mu\nu}F_{\mu\nu}+\bar{q}i\not{D}q+(m\bar{q}_Rq_L+\text{h.c.})+\frac{i\phi}{16\pi^2}\operatorname{Tr}F^{\mu\nu}\tilde{F}_{\mu\nu}\,. \tag{2.14}$$

In this case, the mass parameter is supposed to be the bare mass with a fixed ultraviolet cutoff. These lagrangians describe exactly the same low-energy physics. The situation with the 1PI effective action is more complicated, requiring a detailed discussion on how the low-energy effective θ parameter is related to the bare θ-parameter [18]. However, one does not need to worry about subtleties concerning infrared effects as long as one is dealing with the change of bare parameters needed to keep the physics fixed as the ultraviolet cutoff is varied, because the modes beneath the cutoff are still to be integrated over. Even though the bare parameters are not as directly related to physical observables as those in 1PI effective actions, making exact statements about the physical equivalence of bare lagrangians such as (2.13) and (2.14) is crucial in many applications: e.g. the determination of the effective lagrangians for SUSY YM and SUSY QCD theories in [19].

The fact that we do not refer to the 1PI effective action is desirable. In non-abelian gauge theories, the infrared effects are so severe that 1PI effective action cannot be defined without a clear prescription for an infrared cutoff. In fact, it is not clear what the correct definition of the running gauge coupling constant is in 1PI effective actions. One obvious choice is to use dimensional regularization (or dimensional reduction), which regularizes both the ultraviolet and infrared, possibly with minimal subtraction. Dimensional regularization, however, is not desirable for our purposes precisely because it regularizes both the ultraviolet and infrared divergences; it is impossible to only move the ultraviolet cutoff while leaving the infrared cutoff fixed, and hence it is hard to disentangle different effects. Actually, there is no rescaling anomaly when dimensional regularization is used (see appendix A.1). The two-loop contribution to the β-function of gauge coupling constant, which we describe as a consequence of the rescaling anomaly, appears from infrared uncertain terms $\sim 0/0$ [30] in perturbative calculations, when dimensional regularization is used. This let the authors of refs. [31, 7] claim that the β-function beyond one-loop arises from the infrared in SUSY YM and supersymmetric QED. However, it is not clear from this argument that the two-loop contribution is due to infrared effects, since dimensional regularization mixes up infrared and ultraviolet effects. In fact, in the method of differential renormalization [32], it is clear that conventional β-functions come only from short distance divergences, and the method reproduces standard results for the 2-loop β-functions of Wess-Zumino model [33] and supersymmetric QED [34].

It is nevertheless interesting to ask how the bare coupling constant is related to gauge coupling constant in 1PI effective actions. Recall first that the coupling constants in 1PI effective actions are highly scheme dependent (even gauge-dependent). In order to relate the wilsonian coupling (holomorphic or canonical) to the 1PI coupling, the renormalization scheme must be completely specified. We cannot make a general statement relating wilsonian and 1PI couplings. However, we expect that the canonical gauge coupling is closely related to the 1PI coupling. For instance, one can define the 1PI coupling $g_{\rm 1PI}(q^2)$ by the gauge field two-point function at a fixed euclidean momentum transfer q^2 within the background field formalism. By changing the cutoff down to a scale very close to q^2, one can minimize the difference between the bare lagrangian and the classical 1PI effective action for external fields of momentum $O(q^2)$, since the path integral over quantum fields generates little correction. Therefore the 1PI coupling should be very close to the canonical coupling, $g_{\rm 1PI}(q^2) \approx g_c(q^2)$. If one starts with the bare action with holomorphic normalization, the path integral over quantum fields is not trivial even when $M^2 \approx q^2$, because they do not have canonical normalization; the path integral yields the difference between the holomorphic and canonical coupling, and hence one obtains the same result as in the canonical case. Even though the argument in this paragraph is certainly not rigorous, it does suggest the 1PI coupling is related to g_c rather than g_h. Calling g_c the "1PI coupling" may not be incorrect; the statement about an exact β-function is, however, somewhat empty unless a renormalization scheme is specified. There may also be non-perturbative corrections from the path integral which cannot be seen from this type of perturbative calculation.

3. Anomaly multiplet

One of the confusing points relating to β-functions in $N = 1$ pure SUSY YM is the so-called anomaly multiplet puzzle. At the classical level, the $\mathrm{U}(1)_R$ current belongs to the same multiplet as the energy-momentum tensor (the supercurrent multiplet) [2].

Their anomalous divergences also form the chiral "anomaly multiplet" [3, 4], whose F component is nothing but

$$F = \frac{2}{3}\theta^{\mu}_{\mu} + i\partial_{\mu} j^{\mu}_{R}\,; \tag{3.1}$$

holomorphy relates the $\mathrm{U}(1)_R$ and trace anomalies.

The supersymmetric extension of Adler-Bardeen theorem states that the anomaly of $\mathrm{U}(1)_R$ is exhausted at one-loop [5]. On the other hand, the trace of the energy momentum-tensor is proportional to the β-function of the gauge coupling constant, and hence receives all order contributions. This has been referred to as the anomaly puzzle in supersymmetric gauge theories. Grisaru, Milewski and Zanon [20] studied this question in detail and found that there are two different definitions of the supercurrent. One definition has anomaly exhausted at one-loop and belongs to the same multiplet as the Adler-Bardeen $\mathrm{U}(1)_R$ anomaly; the other has anomaly from all orders in perturbation theory and is proportional to the β-function. The two operators were defined by regularization via dimensional reduction, and differ in the ϵ dimensions. Even though this could well be the resolution of the puzzle, the discussion is highly technical, and the physical meaning of the two operators is not clear. Shifman and Vainshtein [7] also presented a solution to the anomaly puzzle. In their interpretation, the operator equation for the anomalies are indeed exhausted at 1-loop. The all-order contribution to the trace anomaly comes from infrared singularities which arise upon taking the matrix element of the operator relations. However, having understood the NSVZ β function in a purely wilsonian framework with no reference to infrared physics in the previous section, we would now like to address the anomaly puzzle in our framework.

In our language, the resolution to the puzzle is very simple. The anomaly under the $\mathrm{U}(1)_R$ transformation and dilation are related by holomorphy. As long as one maintains the manifest holomorphy, they have anomalies in the same multiplet, and are exhausted at 1-loop. On the other hand, if the vector multiplet has canonical kinetic term, it will not stay canonical after the dilation has been performed, and an additional rescaling is needed in order to go back to canonical normalization. Therefore, this modified dilation (which keeps canonical normalization for the vector multiplet) is no longer related to the $\mathrm{U}(1)_R$ transformation, and its anomaly receives contributions from all orders in perturbation theory according to the NSVZ β function. The two different definitions of the trace of energy momentum tensor are consequences of the two different dilation transformations: the naive one appropriate in holomorphic normalization and the modified one which is designed to preserve canonical normalization. We do not work out the explicit forms of the energy momentum tensor here; instead we explain what dilation transformations are appropriate for the two different normalizations of the vector multiplet and explain how their anomalies naturally differ.

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The anomalous jacobian under the dilation is worked out in appendix B. It is given by

$$\mathcal{D}(V_h(e^{-t}x, e^{-t/2}\theta, e^{-t/2}\bar{\theta})) = \mathcal{D}V_h(x,\theta,\bar{\theta}) \times \\ \times \exp\left(\frac{1}{16}t\int d^2\theta \frac{-3t_2(A)}{8\pi^2}WW + \text{h.c.} + O\left(\frac{1}{M^4}\right)\right), \tag{3.2}$$

and the F-terms are exact just as was the case with the rescaling anomaly. This jacobian adds to the gauge kinetic term and changes the gauge coupling constant. It is nothing but the required change in the holomorphic gauge coupling constant under the change of the cutoff,

$$\frac{1}{g_h^2} \to \frac{1}{g_h^2} + \frac{b_0}{8\pi^2}t\,, \tag{3.3}$$

as $b_0 = -3t_2(A)$ in $N = 1$ pure Yang-Mills theory. The exactness of (the F-term in) the anomalous jacobian is in one-to-one correspondence to the one-loop nature of the running of the holomorphic gauge coupling constant. It is also clear that the jacobians under $\mathrm{U}(1)_R$ and dilation are given in a holomorphic way, both proportional to the WW operator. This is nothing but the anomaly multiplet structure, namely that the divergence of the $\mathrm{U}(1)_R$ current and the trace of the energy momentum tensor are both given by $\int d^2\theta WW$ operator. An explicit regularization method which preserves the manifest holomorphy between $\mathrm{U}(1)_R$ and trace anomalies will be presented in section 6.

The resolution to the anomaly puzzle is the normalization of the vector multiplet. Under the dilation, the gauge kinetic term receives an additional contribution from the jacobian. When one employs holomorphic normalization for the vector multiplet, this is the correct dilation, and no further steps are necessary. On the other hand, starting with a canonically normalized vector multiplet, the additional contribution to the quantum action from the anomalous jacobian (3.2) takes the vector multiplet out of canonical normalization. Therefore, one needs to rescale the vector multiplet to go back to canonical normalization, and this produces another anomalous jacobian. We have the following sequence for the change in the gauge kinetic term. Starting with gauge kinetic term in canonical normalization:

$$\frac{1}{16}\int d^2\theta \frac{1}{g_c^2}W^a(g_cV_c)W^a(g_cV_c)\,, \tag{3.4}$$

the dilation generates an additional contribution to the kinetic term, yielding

$$\frac{1}{16}\int d^2\theta \left(\frac{1}{g_c^2} + \frac{b_0}{8\pi^2}t\right) W^a(g_cV_c)W^a(g_cV_c)\,. \tag{3.5}$$

But now the vector multiplet is no longer canonically normalized. A modified dilation for the vector multiplet which would keep it in canonical normalization is not only the transformation defined above but further requires a rescaling of the vector multiplet. The change of variable $g_c V_c = g'_c V'_c$ produces an additional jacobian as in the previous section, and the gauge kinetic term becomes

$$\frac{1}{16}\int d^2\theta \left(\frac{1}{g_c^2} + \frac{3t_2(A)}{8\pi^2}t - 2\frac{t_2(A)}{8\pi^2}\ln\frac{g'_c}{g_c}\right) W^a(g'_c V'_c) W^a(g'_c V'_c)\,. \tag{3.6}$$

Since this modified dilation

$$V'_c(x,\theta,\bar\theta) = \frac{g_c}{g'_c} V_c(e^{-t}x, e^{-t/2}\theta, e^{-t/2}\bar\theta) \tag{3.7}$$

includes the rescaling of the vector multiplet, it is not in the same anomaly multiplet as the $\mathrm{U}(1)_R$ transformation. Now, g'_c must be chosen so that the coefficient of the WW operator becomes $1/g'^2_c$, giving

$$\frac{1}{g'^2_c} = \frac{1}{g_c^2} + \frac{3t_2(A)}{8\pi^2}t - \frac{2t_2(A)}{8\pi^2}\ln\frac{g'_c}{g_c}\,. \tag{3.8}$$

Of course, in performing the dilation, the cutoff is changed from M to $M' = e^t M$, so $1/g'^2_c$ is the canonical bare coupling needed at cutoff M' in order to keep the physics fixed. Eq. (3.8) then gives the relationship between g_c and g'_c

$$\left(\frac{1}{g'^2_c} + \frac{2t_2(A)}{8\pi^2}\ln g'_c\right) = \left(\frac{1}{g_c^2} + \frac{2t_2(A)}{8\pi^2}\ln g_c\right) - \frac{3t_2(A)}{8\pi^2}\ln\frac{M}{M'}\,, \tag{3.9}$$

which is just eq. (2.11), and the NSVZ β function follows as in eq. (2.12).

4. Models with matter fields

In the case with matter multiplets, the holomorphic lagrangian at cutoff M is

$$\mathcal{L}_h^M(V_h,\phi) = \frac{1}{16}\int d^2\theta \frac{1}{g_h^2} W^a(V_h) W^a(V_h) + \text{h.c.} + \int d^4\theta \sum_i \phi_i^\dagger e^{2V_h^i}\phi_i\,, \tag{4.1}$$

where i runs over the chiral multiplets and $V_h^i = V_h^a T_i^a$ (T_i^a are the generators in the i representation). There are hidden parameters in the above lagrangian, the coefficients Z_i of the kinetic terms for the chiral multiplets. However, we have made the conventional choice and set $Z_i = 1$ in the bare lagrangians. Now, as we change the cutoff from M to M', how must the couplings change in order to keep the low energy physics fixed? Exactly the same argument as in the section 2 shows that, as long as the change in $1/g_h^2$ is holomorphic, this change is exhausted at 1-loop. However, the change can only be holomorphic if we allow the coefficient of the matter kinetic terms (which are manifestly non-holomorphic, being only a function of $\mathrm{Re}(1/g_h^2)$) to

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change from 1 to $Z_i(M,M')$, so that the lagrangian at cutoff M' is

$$\begin{aligned}\mathcal{L}_h^{M'}(V_h,\phi) = \frac{1}{16}\int d^2\theta\left(\frac{1}{g_h^2}+\frac{b_0}{8\pi^2}\ln\frac{M}{M'}\right)W^a(V_h)W^a(V_h)+\text{h.c.}\\ +\int d^4\theta\sum_i Z_i(M,M')\phi_i^\dagger e^{2V_h^i}\phi_i\,,\end{aligned} \tag{4.2}$$

where $b_0=-3t_2(A)+\sum_i t_2(i)$. If we insist on working with canonically normalized kinetic terms for the matter fields, we need to make the change of variable $\phi_i' = Z_i(M,M')^{-1/2}\phi_i$. As with the vector multiplet, however, the measure is not invariant under this change, and there is an anomalous jacobian [13]. In our case, $Z_i(M,M')$ is real, but it is sensible to look at $\mathcal{D}(Z^{-1/2}\phi')$ for a general complex Z since ϕ' is a chiral superfield. Note that when $Z=e^{i\alpha}$ is a pure phase, the change of variable is just a phase rotation of all the components of ϕ, and the jacobian is just the one associated with the chiral anomaly. This jacobian is exactly known and cutoff independent

$$\mathcal{D}(e^{-i\alpha/2}\phi')\mathcal{D}(e^{+i\alpha/2}\bar\phi') = \mathcal{D}\phi'\mathcal{D}\bar\phi'\,\exp\left(\frac{1}{16}\int d^4y\int d^2\theta\frac{t_2(\phi)}{8\pi^2}\ln(e^{i\alpha})WW+\text{h.c.}\right). \tag{4.3}$$

In the case where Z is a general complex number, the cutoff independent piece of this jacobian has been calculated in a manifestly supersymmetric way by Konishi and Shizuya [13], and we present a less technical derivation using components in the appendix. In general the jacobian has both F and D terms. The F terms are known exactly and are the same as in the above with $\ln e^{i\alpha}$ replaced by $\ln Z$. The D terms (such as $\mathrm{Re}(\ln Z)\bar W\bar W WW$) are all suppressed by powers of the cutoff, and can be truly neglected for the same reason as given in the section 2: the non-renormalization theorem makes it impossible for these operators to contribute to F terms.

Therefore, if we wish to work with canonically normalized matter fields at all cutoffs, the lagrangian at cutoff M' must be

$$\mathcal{L'}_h^{M'}(V_h,\phi) = \frac{1}{16}\int d^2\theta\frac{1}{g_h'^2}W^a(V_h)W^a(V_h)+\text{h.c.}+\int d^4\theta\sum_i\phi^\dagger e^{2V_h^i}\phi\,, \tag{4.4}$$

with

$$\frac{1}{g_h'^2} = \frac{1}{g_h^2}+\frac{b_0}{8\pi^2}\ln\frac{M}{M'}-\sum_i\frac{t_2(i)}{8\pi^2}\ln Z_i(M,M')\,. \tag{4.5}$$

If we now wish to further work with canonical kinetic terms for the vector multiplet, we need to rescale the vector field as in the previous section, with the same result. The combination

$$\frac{1}{g_c^2}+\frac{2t_2(A)}{8\pi^2}\ln g_c+\sum_i\frac{t_2(i)}{8\pi^2}\ln Z_i \tag{4.6}$$

runs only at 1-loop, and the NSVZ β function (1.1) follows trivially

$$\mu\frac{dg_c}{d\mu} = -\frac{g_c^3}{8\pi^2}\frac{3t_2(A) - \sum_i t_2(i)(1-\gamma_i)}{1 - t_2(A)g_c^2/8\pi^2}\,, \tag{4.7}$$

where $\gamma_i = (\mu d/d\mu)\ln Z_i(M,\mu)$.

5. $N = 2$ theories

We now turn to the analysis of $N = 2$ theories, which are known to be finite above 1-loop [22]. Here we will explain this result by using the anomalous jacobians we have derived. We can also use the known finiteness of these theories above 1-loop, proved perturbatively, to give an alternate derivation of the jacobian for the rescaling of the vector multiplet, which we used to derive the Shifman-Vainshtein formula (2.10).

Using $N = 1$ language, the holomorphic lagrangian for pure $N = 2$ supersymmetric Yang-Mills theories is

$$\mathcal{L}(V_h,\phi_h) = \frac{1}{16}\int d^2\theta \frac{1}{g_h^2} W^a(V_h)W^a(V_h) + \text{h.c.} + \int d^4\theta\, \mathrm{Re}\left(\frac{2}{g_h^2}\right)\mathrm{Tr}\,\phi_h^\dagger e^{-2V_h}\phi_h e^{2V_h}\,, \tag{5.1}$$

where ϕ_h is a chiral field in the adjoint representation of the gauge group. As discussed in the previous sections, the holomorphic gauge coupling only changes at 1-loop when we change the cutoff from M to M'. The coefficients of the kinetic terms of the $N = 1$ vector multiplet and the adjoint field are both changed according to the holomorphic gauge coupling as required by $N = 2$ invariance. If we wish to work with canonically normalized fields, we must make the change of variables $\phi_h = g_c\phi_c, V_h = g_c V_c$ (where the rescalings for ϕ, V must be the same by $N = 2$ supersymmetry). We can compute the jacobian for this variable change from the jacobians for matter and vector fields we found in the previous two sections. The final result is that the jacobian for the vector multiplet cancels the one from the adjoint chiral multiplet:

$$\begin{aligned}\mathcal{D}(g_cV_c) &= \mathcal{D}V_c\, \exp\left(\frac{1}{16}\int d^4y\int d^2\theta \frac{2t_2(A)}{8\pi^2}\ln g_c W^a(g_cV_c)W^a(g_cV_c) + \right.\\ &\qquad\left. + \text{h.c.} + \mathcal{O}\left(\frac{1}{M^4}\right)\right)\,,\\ \mathcal{D}(g_c\phi_c) &= \mathcal{D}\phi_c\, \exp\left(\frac{1}{16}\int d^4y\int d^2\theta - \frac{2t_2(A)}{8\pi^2}\ln g_c W^a(g_cV_c)W^a(g_cV_c) + \right.\\ &\qquad\left. + \text{h.c.} + \mathcal{O}\left(\frac{1}{M^4}\right)\right)\,.\end{aligned} \tag{5.2}$$

Therefore the canonical coupling coincides with the holomorphic one, and so pure $N = 2$ theories must be perturbatively finite above 1-loop. When $N = 2$ hypermultiplets are added to the theory, the β function still vanishes above 1-loop, since the kinetic terms for the hypermultiplets are not renormalized [23], so there is no need to rescale them to go back to canonical normalization.

As already mentioned, we can turn around the above arguments. Since the finiteness of $N = 2$ theories beyond 1-loop has been explicitly established in perturbation theory, it must be that the canonical coupling coincides with the holomorphic coupling for these theories, which in turn means that $\mathcal{D}(g_c V_c)\mathcal{D}(g_c\phi_c) = \mathcal{D}V_c\mathcal{D}\phi_c$. However by holomorphy, the jacobian for $\mathcal{D}(g_c\phi_c)$ can be inferred from the chiral anomaly jacobian $D(e^{i\alpha}\phi_c)$ without computation. From this, we can deduce the jacobian for the vector multiplet, and therefore the Shifman-Vainshtein formula, as we did in section 3.

6. Regularization

In all the above analysis, we have somewhat loosely been referring to the theory defined with a cutoff M, without defining how the theory is to be cut off. This problem is also related to the question of the scheme in which the Shifman-Vainshtein formula, and hence the NSVZ β-function, holds. We address these questions in this section, by explicitly regulating some $N = 1$ theories using finite $N = 4$ and $N = 2$ theories. We will then give an explicit definition of $1/g_h^2(M)$ and $1/g_c^2(M)$, and will show that they are related by the Shifman-Vainshtein formula.

The idea is very simple. Let us begin with theories with $N = 4$ supersymmetry, which are known to be finite [24]. In $N = 1$ language, these theories contain 1 vector multiplet V and 3 chiral multiplets ϕ^i in the adjoint representation. Now, suppose we add a mass term $\int d^2\theta M \,\mathrm{Tr}(\phi^i\phi^i) + \text{h.c.}$ to the adjoints. Since this is a soft breaking of $N = 4$ SUSY, the theory is still free of UV divergences [25]. Beneath the scale M, it looks like pure $N = 1$ SYM. Thus, we have an explicit regularization for pure $N = 1$ SUSY YM with a cutoff M, and the cutoff moreover preserves manifest holomorphy.

To be specific, we define the holomorphic pure $N = 1$ SUSY YM theory, regulated with cutoff M_h and with gauge coupling $1/g_h^2(M)$, by the lagrangian

$$\begin{aligned}\mathcal{L}_h^{M_h}(V_h, \phi_h^i) = &\frac{1}{16}\int d^2\theta \frac{1}{g_h^2(M_h)} W^a(V_h)W^a(V_h) + \text{h.c.} + \\ &+ \int d^4\theta \,\mathrm{Re}\left(\frac{2}{g_h^2(M_h)}\right) \mathrm{Tr}\left(\phi_h^{i\dagger} e^{-2V_h}\phi_h^i e^{2V_h}\right) + \\ &+ \int d^2\theta \,\mathrm{Re}\left(\frac{1}{g_h^2(M_h)}\right)\sqrt{2}\,\mathrm{Tr}\left(\phi_h^i[\phi_h^j, \phi_h^k]\right)\frac{\epsilon_{ijk}}{3!} + M_h \,\mathrm{Tr}(\phi_h^i\phi_h^i) + \text{h.c.}\end{aligned} \tag{6.1}$$

Since this theory is finite, its ultraviolet cutoff is taken to be infinite. The coupling $1/g_h^2(M_h)$ is the holomorphic coupling of the theory with the infinite ultraviolet cutoff and is finite. On the other hand, we are interested in this theory as the regularized $N = 1$ SUSY YM with ultraviolet cutoff M_h. The holomorphic coupling $1/g_h^2(M_h)$ can be specified independently of M_h; we specify M_h as the argument, however, because later in eq. (6.3) we will vary M_h and $1/g_h^2(M_h)$ at the same time keeping correlation functions of the $N = 1$ SUSY YM fixed.

We similarly define the theory with canonically normalized kinetic terms for the vector multiplet, regulated with cutoff M_c and with gauge coupling $1/g_c^2$ by

$$\begin{aligned}\mathcal{L}_c^{M_c}(V_c, \phi_c^i) = & \frac{1}{16}\int d^2\theta \frac{1}{g_c^2(M_c)} W^a(g_c(M_c)V_c)W^a(g_c(M_c)V_c) + \text{h.c.} \\ & + \int d^4\theta\, 2\,\mathrm{Tr}\left(\phi_c^{i\dagger} e^{-2g_c(M_c)V_c}\phi_c^i e^{2g_c(M_c)V_c}\right) + \\ & + \int d^2\theta\, \sqrt{2} g_c(M_c)\,\mathrm{Tr}\left(\phi_c^i[\phi_c^j, \phi_c^k]\right)\frac{\epsilon_{ijk}}{3!} + M_c\,\mathrm{Tr}(\phi_c^i\phi_c^i) + \text{h.c.}\end{aligned} \tag{6.2}$$

where the relative normalizations of all the terms in the above are chosen to ensure $N = 4$ supersymmetry in the $M_c \to 0$ limit.

The first thing we have to check is that, as M_h is changed, the holomorphic coupling in the $N = 4$ lagrangian must be changed according to the one-loop law to keep the low-energy physics fixed. Requiring that correlation functions of low-energy fields do not vary as M_h is changed, we have[7]

$$M_h \frac{d}{dM_h}\int \mathcal{D}V_h \prod_{i=3}^{3}\mathcal{D}\phi_h^i e^{-S}\mathcal{O}_1\cdots\mathcal{O}_n = 0\,, \tag{6.3}$$

and its complex conjugate equation in terms of $\bar{M}_h$. Here and below, $\mathcal{O}_i$ are arbitrary operators of V_h, and $S = \int d^4x \mathcal{L}_h^{M_h}(V_h, \phi_h^i)$. The change required for $1/g_h^2(M)$ is therefore determined by the equation

$$\int \mathcal{D}V_h \prod_{i=3}^{3}\mathcal{D}\phi_h^i e^{-S}\mathcal{O}_1\cdots\mathcal{O}_n \left\{\left(M_h\frac{d}{dM_h}\frac{1}{g_h^2(M_h)}\right)S_0 + \int d^4x d^2\theta M_h\,\mathrm{Tr}\,\phi_h^i\phi_h^i\right\} = 0\,, \tag{6.4}$$

where S_0 is the action without the mass term and with the overall $1/g_h^2(M_h)$ dropped. Recall that the operators $\mathcal{O}_i$ of physical interest do not involve the regulator fields ϕ^i. Therefore, we can replace the $M_h\,\mathrm{Tr}\,\phi_h^i\phi_h^i$ operator in the above matrix element by operators of the low-energy fields, i.e. W_α, in the sense that any correlation function of the above operator $\int d^4x d^2\theta M_h\,\mathrm{Tr}\,\phi^i\phi^i$ with other operators of low-energy fields can be given, through a systematic expansion in $1/M_h$, by operators which involve

[7] Up to corrections suppressed by powers in $1/M_h$.

low-energy fields only.[8] Such an expansion can be done easily[9] and yields

$$\left\langle \mathcal{O}_1 \cdots \mathcal{O}_n \left(M_h \frac{d}{dM_h} \frac{1}{g_h^2(M_h)} \right) \frac{1}{16} \int d^4x d^2\theta WW \right\rangle =$$

$$= - \left\langle \mathcal{O}_1 \cdots \mathcal{O}_n \int d^4x d^2\theta M_h \operatorname{Tr} \phi_h^i \phi_h^i \right\rangle$$

$$= - \left\langle \mathcal{O}_1 \cdots \mathcal{O}_n \left(\frac{1}{16} \int d^4x d^2\theta \frac{-3t_2(A)}{8\pi^2} WW + \right.\right.$$

$$\left.\left. + \text{higher-dimensional } D\text{-terms of } O\left(\frac{1}{M^4}\right)\right)\right\rangle , \quad (6.5)$$

where we dropped operators such as $\operatorname{Tr} \bar{\phi} e^{2V} \phi e^{-2V}$ in S_0 which only produce contributions suppressed by M_h. In fact, the F-term WW in the above equality can be shown to be exact using the instanton argument given in the appendix. The higher-dimensional D-terms can be dropped without modifying relevant couplings in the bare lagrangian as discussed in section 2. By combining eqs. (6.4) and (6.5), we find that the low-energy physics can be kept fixed by changing the holomorphic gauge coupling according to the one-loop law

$$M_h \frac{d}{dM_h} \frac{1}{g_h^2(M_h)} = \frac{3t_2(A)}{8\pi^2} , \quad (6.6)$$

when one changes M_h. This explicit calculation verifies the exact one-loop law for the change of holomorphic coupling derived indirectly from the argument based on holomorphy given in section 2: given a regularization preserving holomorphy, the running of $1/g_h^2$ is *guaranteed* to be exhausted at 1-loop.

Note that it is only the mass term which breaks both the conformal symmetry and the non-anomalous $\mathrm{U}(1)_R$ symmetry under which all three ϕ^i have charge 2/3. Therefore the response under dilation and $\mathrm{U}(1)_R$ transformations are described the same matrix element. This is the anomaly multiplet structure:

$$\frac{1}{2}\left(\theta_\mu^\mu + i\frac{3}{2}\partial_\mu j_R^\mu \right) = \int d^2\theta M_h \operatorname{Tr}(\phi_h^i \phi_h^i) , \quad (6.7)$$

where the right hand side can be replaced by $\frac{1}{16}\int d^2\theta \frac{-3t_2(A)}{8\pi^2} WW$ in the zero-momentum limit as in eq. (6.5). These are indeed the correct trace and $\mathrm{U}(1)_R$ anomalies for $N=1$ pure SUSY YM in holomorphic normalization.

[8] In other words, we take the expectation value of $M_h \operatorname{Tr} \phi_h^i \phi_h^i$ within $\mathcal{D}\phi_h^i$ path integral in the background gauge field V_h, expanding in powers of $1/M_h$. This is the valid procedure because none of the operators $\mathcal{O}_i$ depend on ϕ_h^i and are outside the $\mathcal{D}\phi_h^i$ path integral.

[9] This calculation is identical to the derivation of the chiral anomaly [26] or the Konishi anomaly [13] with Pauli-Villars regularization, the only difference being the opposite statistics of the ϕ_i relative to the Pauli-Villars regulator fields.

Now that we have defined what we mean by $1/g_h^2(M), 1/g_c^2(M)$, we can relate them to each other. We want to make the change of variable $\phi_h^i = g_c\phi_c^i, V_h = g_cV_c$. From the previous section, we know that the jacobian of V cancels the one from one adjoint multiplet, but this leaves the jacobian for 2 adjoint multiplets left uncancelled. Therefore,

$$\mathcal{D}(g_cV_c)\prod_{i=1}^{3}\mathcal{D}(g_c\phi_c^i) = \mathcal{D}V_c\prod_{i=1}^{3}\mathcal{D}\phi_c^i \exp\biggl(-2\times\frac{1}{16}\int d^4y\int d^2\theta\frac{2t_2(A)}{8\pi^2}\times \\ \times\ \ln g_c W^a(g_cV_c)W^a(g_cV_c) + \text{h.c.} + \cdots\biggr), \tag{6.8}$$

where $\cdots$ refers to the extra terms needed to make the jacobian $N=4$ supersymmetric (see appendix A.5). Note that there are no higher dimension D terms in the above jacobian: since the $N=4$ theory is finite, the cutoff used in computing the jacobian can be taken to infinity with no difficulty. Therefore, we do not need to use a non-renormalisation argument to justify dropping the D terms in the jacobian, as they are simply absent.

Using this result, we find

$$\begin{aligned} Z &= \int\mathcal{D}V_h\prod_i\mathcal{D}\phi_h^i\times \\ &\quad\times\exp\biggl(-\frac{1}{16}\int d^4y\int d^2\theta\frac{1}{g_h^2(M_h)}W^a(V_h)W^a(V_h) + M_h\,\mathrm{Tr}(\phi_h^i\phi_h^i) + \text{h.c.} + \cdots\biggr) \\ &= \int\mathcal{D}(g_cV_c)\prod_i\mathcal{D}(g_c\phi_c^i)\times \\ &\quad\times\exp\biggl(-\frac{1}{16}\int d^4y\int d^2\theta\frac{1}{g_h^2(M_h)}W^a(g_cV_c)W^a(g_cV_c) \\ &\qquad + g_c^2M_h\,\mathrm{Tr}(\phi_c^i\phi_c^i) + \text{h.c.} + \cdots\biggr) \\ &= \int\mathcal{D}V_c\prod_i\mathcal{D}\phi_c^i\times \\ &\quad\times\exp\biggl(-\frac{1}{16}\int d^4y\int d^2\theta\left(\frac{1}{g_h^2(M_h)} + \frac{4t_2(A)}{8\pi^2}\ln g_c\right)W^a(g_cV_c)W^a(g_cV_c) \\ &\qquad + g_c^2M_h\,\mathrm{Tr}(\phi_c^i\phi_c^i) + \text{h.c.} + \cdots\biggr). \end{aligned} \tag{6.9}$$

Defining $M_c = M_hg_c^2$, we must have

$$\frac{1}{g_c^2(M_c)} = \mathrm{Re}\left(\frac{1}{g_h^2(M_h = M_c/g_c^2)}\right) + \frac{4t_2(A)}{8\pi^2}\ln g_c(M_c)\,. \tag{6.10}$$

Using the 1-loop law for $1/g_h^2$,

$$\frac{1}{g_h^2(M_c/g_c^2)} = \frac{1}{g_h^2(M_c)} - \frac{3t_2(A)}{8\pi^2}\ln g_c^2(M_c)\,, \tag{6.11}$$

we finally have

$$\frac{1}{g_c^2(M_c)} = \mathrm{Re}\left(\frac{1}{g_h^2(M_c)}\right) - \frac{2t_2(A)}{8\pi^2}\ln g_c(M_c)\,, \tag{6.12}$$

which is precisely the Shifman-Vainshtein formula (2.10). Because the holomorphic coupling has already been shown to run only at one-loop, the canonical coupling in this explicit regularization follows the NSVZ β-function.

One can repeat exactly the same exercise using finite $N = 2$ theories. In $N = 1$ language, these theories contain the vector multiplet V and a chiral multiplet ϕ in the adjoint representation forming the pure $N = 2$ vector multiplet, as well as vector-like pairs of chiral fields $Q_i, \tilde{Q}_i$, chosen so the 1-loop β function vanishes

$$b_0 = 3t_2(A) - t_2(A) - \sum_i t_2(i) = 0\,. \tag{6.13}$$

Suppose we wish to regulate an $N = 1$ theory with the multiplets $Q_i, \tilde{Q}_i$ (an example would be SUSY QCD with $2N$ flavors). This can be done by starting with the $N = 2$ theory with a mass term $M_h \,\mathrm{Tr}\,\phi_h^2$ added to the adjoint, which preserves the finiteness of the theory [27]. If we now wish to go back to canonical normalization for the gauge kinetic terms, we make the change $V_h = g_c V_c$, $\phi_h = g_c \phi_c$. As mentioned previously, in this case contributions from the vector and chiral multiplets cancel in the jacobian. However, the mass term for the adjoint becomes $M_c\phi_c^2 = M_h g_c^2 \phi_c^2$, and so

$$\frac{1}{g_c^2(M_c)} = \mathrm{Re}\left(\frac{1}{g_h^2(M_h = M_c/g_c^2)}\right). \tag{6.14}$$

Using the 1-loop law for $1/g_h^2$ together with $3t_2(A) - \sum_i t_2(i) = t_2(A)$, we again find the Shifman-Vainshtein formula

$$\begin{aligned}\frac{1}{g_c^2(M_c)} = \mathrm{Re}\left(\frac{1}{g_h^2(M_c/g_c^2)}\right) &= \mathrm{Re}\left(\frac{1}{g_h^2(M_c)}\right) - \frac{3t_2(A) - \sum_i t_2(i)}{8\pi^2}\ln g_c^2(M_c) \\ &= \mathrm{Re}\left(\frac{1}{g_h^2(M_c)}\right) - \frac{2t_2(A)}{8\pi^2}\ln g_c(M_c)\,. \end{aligned}\tag{6.15}$$

In the case where the hypermultiplets are $2N$ flavors of an $\mathrm{SU}(N)$ gauge group, we can add mass terms to some of the hypermultiplets, thereby regularizing an arbitrary $N = 1$ $\mathrm{SU}(N)$ with $N_f < 2N$ flavors. Similar considerations lead to the Shifman-Vainshtein relation and the NSVZ β function in this case as well.

We obviously cannot extend the regularization methods discussed in this section to general $N = 1$ theories, especially chiral ones. However, our observation that an explicit regularization preserving holomorphy and yielding NSVZ β function exists for a class of $N = 1$ theories does support our argument. One can hope that a certain regularization is possible for general $N = 1$ theories which preserves manifest holomorphy, perhaps by higher-derivative regularization for the vector multiplet [28] and the infinite tower of Pauli-Villars regulators for chiral multiplets [29].

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It is noteworthy that the exact NSVZ β-function can be checked by explicit perturbative calculations, since we have given an explicit regularization scheme for $N = 1$ SUSY YM. The procedure will be as follows. One works out certain Green's functions (e.g. gauge field two-point function in background field method) as a function of external momenta, bare coupling and the regulator mass M. Then one tries to change the cutoff and the bare coupling at the same time to keep the Green's function fixed. In this way, the correct β-function for the wilsonian coupling constant can be determined. Since the theory is finite, there should be no ambiguity in the analysis as long as the Green's function under study is free from infrared singularities.

7. Conclusions

In this paper, we hope to have clarified some of the mysteries surrounding the gauge coupling β functions for SUSY gauge theories. The result is quite simple: if we work with the holomorphic bare lagrangian with a cutoff M, the change in $1/g_h^2$ needed to keep the low energy physics fixed as the cutoff is changed from M to M' is exhausted at 1-loop. However, since the rescaling of the vector multiplet in going to canonical normalization for the matter fields is anomalous, the gauge coupling g_c in the theory with canonical kinetic terms is not equal to g_h. The F terms in this jacobian can be determined exactly, while the D terms are suppressed by powers of the cutoff. In a non-supersymmetric theory, these higher dimension operators can in general feed back in to the coefficient of relevant operators at higher orders; however in our case the higher dimension D terms are forbidden from doing so by the non-renormalization theorem. The final relationship between g_c and g_h is given by the Shifman-Vainshtein formula (2.10), and the change in g_c upon moving the cutoff from M to M' is given by the NSVZ β function (2.12). Our analysis does not encounter any subtleties from infrared physics because we never refer to 1PI effective actions. All the discussions are on bare couplings within the framework of wilsonian effective action with a regularization in the ultraviolet. This is desirable because we can make a separation between the ultraviolet structure of the theories (which determine the evolution of the couplings) and model-dependent, dynamical effects from infrared singularities. We have understood that $N = 2$ theories have only one-loop β-function because the rescaling anomaly cancels between the vector multiplet and adjoint chiral multiplet in $N = 1$ language. Finally, we have shown that certain $N = 1$ theories can be regularized with a cutoff M starting from finite $N = 4$ and $N = 2$ theories, in a way preserving manifest holomorphy. In these theories, we have demonstrated that the Shifman-Vainshtein relation, and hence the NSVZ β-function, holds. The claimed exactness of the β function can then at least in principle be checked by direct calculation in these explicitly regularized theories.

Acknowledgments

We thank Dan Freedman, Bogdan Morariu, Hirosi Ooguri, and Bruno Zumino for many useful discussions and comments on the manuscript. N.A.H. also thanks Hsin-Chia Cheng and Takeo Moroi for discussions. This work was supported in part by the Director, Office of Energy Research, Office of High Energy and Nuclear Physics, Division of High Energy Physics of the U.S. Department of Energy under Contract DE-AC03-76SF00098 and in part by the National Science Foundation under grant PHY-90-21139. N.A.H. was also supported by NSERC, and H.M. by Alfred P. Sloan Foundation.

A. Rescaling anomaly

A.1 Generalities

In this appendix, we discuss various aspects of the anomaly incurred in changing the normalization of fields in the path integral, which we call the rescaling anomaly. The rescaling of a quantum field is simply a change of variables $\phi(x) = e^{\alpha}\phi'(x)$. In a general non-supersymmetric theory, this change of variable is not unitary and is not expected to leave the measure invariant; the jacobian for this transformation has power ultraviolet divergences and is highly regularization dependent. Nevertheless, once a specific choice of regularization is made, one must carefully account for the correct jacobian when making rescalings in the path integral.

In a supersymmetric theory, the situation is different: the transformation $f(x, \theta, \bar{\theta}) = e^{\alpha} f'(x, \theta, \bar{\theta})$ for superfields *does* naively leave the measure $\mathcal{D}f$ invariant, since the jacobians from bosonic and fermionic components naively cancel. Of course, whether or not a non-trivial jacobian exists depends both on what type of regularization is used and the symmetries which need to be preserved. In the case of supersymmetric gauge theories, the preservation of gauge invariance and supersymmetry force a non-trivial jacobian for the rescaling of both chiral and vector multiplets. The calculation of these jacobians is the main purpose of this appendix. First, however, some preliminary remarks are in order.

We usually do not encounter the rescaling anomaly in perturbation theory. The reason is somewhat trivial: by convention, we employ canonical normalization for bare fields and never change the normalization. The wave function renormalization is applied to the fields in the 1PI effective action, where the rescaling of the fields is nothing more than a relabeling of variables, as the 1PI effective action is a classical object and no further functional integration is done. On the other hand, a wilsonian action (i.e. a bare theory defined with its cutoff) retains quantum fields beneath the cutoff, which are then integrated over. If one rescales the fields in a wilsonian effective action, the correct jacobians must be taken into account. The jacobian gives the modification of the bare lagrangian necessary to keep the physics fixed after rescaling the quantum fields, at a fixed value of the cutoff. In principle, with a given cutoff, we just need to compute in the theories before and after the rescaling, and explicitly see what changes are necessary in the bare lagrangian in order to keep all amplitudes fixed.

In practice, however, when we compute jacobians, they are typically regularized by hand in a way that preserves the important symmetries. It is not a priori clear how the regularization of jacobian is related to the way in which the full theory is regularized; in principle, any given regularization of the full theory should specify the regularization of the jacobians. In some cases, there is no problem with being sloppy about this point. For example, in the case of the chiral anomaly, we know that the jacobian is completely topological in nature and is independent of the way in which the theory is regularized (providing the regularization is gauge-invariant). Therefore, directly regularizing the jacobian of a chiral transformation, say by gaussian damping as in the Fujikawa method [36], will give us the exact answer (i.e. it gives us the exact modification of the bare lagrangian needed to keep the physics fixed), since the answer is regulator independent. In other cases, however, we have to be more careful. For instance, in the case of the jacobian for dilation [37], the regulator independent pieces correctly reproduce the 1-loop β function, but where do all the higher order contributions to the β function come from? The answer must be that either the higher dimension operators in the jacobian can not simply be thrown out, or the quartic divergence in the jacobian contains hidden dependence on the fields at higher orders. Recall that the higher-dimension operators can only be set to zero after an appropriate modification of the relevant couplings, presumably providing the higher order corrections [10]. In this case, while we can regularize the jacobian to get the 1-loop β function, the way in which the jacobian is regularized must be *derived* from the regularization of the full theory in order to get the higher order corrections.

Having said this, in this appendix we will regularize all the jacobians we encounter by hand. The reason is that, somewhat analogous to the situation with the chiral anomaly, we can be sloppy about the regularization here, for the following two reasons. First, the jacobians come out automatically finite, and there is no concern about an infinite constant changing the result at higher orders. Second, the F term in the jacobian can be exactly computed and is regularization independent. Third, while the jacobian (regulated by hand) does contain D terms suppressed by powers of the cutoff, we don't need to know the precise way in which this is related to how the full theory is regularized; the usual supersymmetric non-renormalization theorem makes it impossible for these D terms to ever feed back into an F term like WW, and so they are truly irrelevant for our interests.

Another remark is that the dimensional regularization does not produce a rescaling anomaly. This is a consequence of the following identity: $\int d^D p$ const. $= 0$. When one employs dimensional regularization (or more correctly, regularization by dimensional reduction), the would-be effect of the rescaling anomaly appears as a part of conventional perturbation theory. Indeed, in perturbation theory using dimensional regularization, the two-loop contribution to the β-function, which we describe as a consequence of the rescaling anomaly, appears from infrared uncertain terms $\sim 0/0$ [30]. However, this does not necessarily imply that the two-loop contribution is coming from the infrared, since dimensional regularization mixes up infrared and ultraviolet effects.

A.2 Supersymmetric path integrals

Since the vacuum energy vanishes in a supersymmetric background, the path integral around a supersymmetric background is simply unity, and we do not expect any anomalous rescaling jacobian in this case. Let us first see how this works for for an $N = 1$ chiral supermultiplet. The lagrangian is, given in terms of components,[10]

$$\mathcal{L} = \int d^4x e^{-2\alpha} \left(\partial_\mu \bar{\phi} \partial^\mu \phi + \bar{\psi} \partial\!\!\!/ \psi - \bar{F} F\right) + \int d^4x e^{-2\alpha} m \left(\frac{1}{2}\psi\psi + \phi F\right) + \text{h.c.}\,, \quad \text{(A.1)}$$

where we have for convenience included the $e^{-\alpha}$ factor in the lagrangian. Naively, if we just redefine $\phi = e^\alpha \phi'$, and if the measure is invariant, nothing should depend on α. In this trivial free theory, this is certainly the case. The path integral is given by

$$Z = \frac{\det(e^{-2\alpha}(\partial\!\!\!/ + m))}{\det(e^{-2\alpha}(-\Box + m^2)) \det(e^{-2\alpha})} = 1\,. \quad \text{(A.2)}$$

The determinant in the numerator is taken over two-dimensional spinor space, and hence $e^{-2\alpha}$ factor is counted twice. Not surprisingly, the α dependence drops out only when the auxiliary component is included; one cannot simply replace the F-component by its solution to the equation of motion. It is important to keep the $N = 1$ off-shell multiplet structure in path integrals.[11]

In fact, the jacobian for rescaling a chiral superfield can be calculated directly à la Fujikawa without referring to the determinants. The jacobians can be regularized by the kinetic operator. For a massive chiral superfield, one can use the gaussian regularization

$$e^{-t(-L+m^2)}\,, \quad \text{(A.3)}$$

where $t = 1/M^2$ is an ultraviolet cutoff, and $L = \bar{D}^2 D^2/16 = \Box$ when acting on a chiral superfield. The jacobian is trivial because of a supersymmetric cancellation

$$\ln J = \alpha \left(\mathrm{Tr}_\phi\, e^{-t(-\Box+m^2)} - \mathrm{Tr}_\psi\, e^{-t(-\Box+m^2)} + \mathrm{Tr}_F\, e^{-t(-\Box+m^2)}\right) = 0\,. \quad \text{(A.4)}$$

[10] This is a euclidean lagrangian [35]. There is no distinction between upper and lower indices. The spinors ψ and $\bar{\psi}$ are not related by complex conjugation; they must be treated as completely independent. The auxiliary fields F and $\bar{F}$ are also independent, and in fact, it is necessary to rotate their contours from F to iF, $\bar{F}$ to $i\bar{F}$ to make the gaussian integral over F, $\bar{F}$ fields possible. We write lagrangians before the rotation of F, $\bar{F}$ fields so that the correspondence to the Minkowski lagrangian is more clear

[11] We fortunately do not need off-shell multiplets of extended supersymmetry.

Again, the jacobian from the F-component must be included, and in this case cancels the contributions from the scalar and spinor components.

A.3 Chiral multiplets

Here we calculate the rescaling anomaly of a chiral multiplet in background gauge field. This calculation was done first by Konishi and Shizuya [13] using the superfield formalism. We repeat their analysis in terms of component fields in order to gain a better intuition on the anomalous jacobian.

The path integral of a chiral multiplet in a gauge field background is given by

$$\int \mathcal{D}\phi\mathcal{D}\psi\mathcal{D}F\mathcal{D}\bar{\phi}\mathcal{D}\bar{\psi}\mathcal{D}\bar{F}\, e^{-\int d^4x\left(|D\phi|^2+\bar{\psi}\not{D}\psi-\bar{F}F\right)}\,. \tag{A.5}$$

We will discuss the case of Abelian gauge theory with the covariant derivative $D_\mu = \partial_\mu - iA_\mu$, but the extension to non-abelian case is straight-forward. We calculate the anomalous jacobian of the measure $\mathcal{D}\Phi = \mathcal{D}\phi\mathcal{D}\psi\mathcal{D}F$ under the rescaling of the chiral superfield $\Phi(y,\theta) = \phi(y) + \sqrt{2}\theta\psi(y) + \theta^2 F(y)$. Note that we treat ϕ and $\bar{\phi}$ etc independently.

Under the rescaling $\Phi = e^{\alpha}\Phi'$, with α a general complex number, we formally have

$$\mathcal{D}\Phi = \mathcal{D}\phi\mathcal{D}\psi\mathcal{D}F = \mathcal{D}\phi'(\det e^{\alpha})\mathcal{D}\psi'(\det e^{\alpha})^{-2}\mathcal{D}F'(\det e^{\alpha}) = \mathcal{D}\Phi' J\,, \tag{A.6}$$

where all the jacobian factors appear to cancel out. However, we need to regularize the jacobians appropriately:

$$\ln J = \alpha\left(\mathrm{Tr}_\phi\, e^{t(D_\mu)^2} - \mathrm{Tr}_\psi\, e^{t\not{D}^2} + \mathrm{Tr}_F\, e^{t(D_\mu)^2}\right) \tag{A.7}$$

and they may not cancel out exactly as we will see below. Note that the contribution from ψ is a trace over two-component spinor space. The expression is proportional to $(1-2+1)=0$ with trivial background $D_\mu = \partial_\mu$, but gives a non-vanishing result for non-trivial background gauge fields.

The above choice of the gaussian regularization is motivated by the following reason. In the case where α is imaginary, we have chiral rotation on the fermion fields, and the jacobian is the one associated with the chiral anomaly, so the Fujikawa method suggests the usual $\not{D}^2$ gaussian damping. Furthermore, if the fermion fields are expanded in eigenmodes of $\not{D}^2$, their kinetic term is diagonal and the symmetries of the action are manifest. This suggests that the scalar component be damped by its kinetic term D_μ^2. Furthermore, we know that the anomalous jacobians in a trivial background $A_\mu = 0$ must cancel between different components in the same supermultiplet because supersymmetry fixes the normalization of path integrals to unity. Therefore, we must choose a gaussian regularization for the jacobian of the auxiliary component which cancels the anomalous jacobians from the scalar and

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spinor components in the trivial background, and hence it must be $e^{t(\partial_\mu)^2}$ in the trivial background. The only possible gauge covariant extension is $e^{t(D_\mu)^2}$ because the auxiliary component transforms the same way as the other components under the ordinary gauge transformations. We will see later that this regularization of the components can be justified in a manifestly supersymmetric way [13], but the above arguments are a quick route to the correct answer.

For the case of constant background field strength and charge +1 chiral superfield, the traces (heat kernels) can be evaluated explicitly using standard harmonic oscillator methods. For completeness, we review the methods here. (For a manifestly supersymmetric calculation of the heat kernel for a chiral superfield, see [39]). We employ background gauge field with constant electric and magnetic fields, because the cutoff dependence of the result can be explicitly seen without approximations. One can diagonalize the field strength tensor $F_{\mu\nu}$ to the form

$$F_{\mu\nu} = \begin{pmatrix} 0 & E & 0 & 0 \\ -E & 0 & 0 & 0 \\ 0 & 0 & 0 & B \\ 0 & 0 & -B & 0 \end{pmatrix}. \tag{A.8}$$

Let us pick a gauge with $A_1 = -Ex_2/2, A_2 = Ex_1/2, A_3 = -Bx_4/2, A_4 = Bx_3/2$, which clearly reproduces the above $F_{\mu\nu}$.[12] Then, with $p_\mu = -i\partial_\mu$, we have

$$\begin{aligned} \operatorname{Tr} e^{t(D_\mu)^2} &= \operatorname{Tr} e^{-t\left((p_1-\frac{E}{2}x_2)^2+(p_2+\frac{E}{2}x_1)^2+(p_3-\frac{B}{2}x_4)^2+(p_4+\frac{B}{2}x_3)^2\right)} \\ &= \operatorname{Tr} e^{-tH(E)} \operatorname{Tr} e^{-tH(B)}, \end{aligned} \tag{A.9}$$

where

$$\begin{aligned} H(E) &= \left(p_1 - \frac{E}{2}x_2\right)^2 + \left(p_2 + \frac{E}{2}x_1\right)^2 \\ &= p_1^2 + p_2^2 + \left(\frac{E}{2}\right)^2 (x_1^2 + x_2^2) - E(p_1 x_2 - p_2 x_1). \end{aligned} \tag{A.10}$$

Defining the usual harmonic oscillator raising and lowering operators as

$$p_\mu = \sqrt{\frac{2}{E}}\left(\frac{a_\mu - a_\mu^\dagger}{\sqrt{2}i}\right), \qquad x_\mu = \sqrt{\frac{E}{2}}\left(\frac{a_\mu + a_\mu^\dagger}{\sqrt{2}}\right) \tag{A.11}$$

and further defining $a_0 = (a_L + a_R)/\sqrt{2}, a_1 = (a_L - a_R)/\sqrt{2}i$, we find $H(E) = E(2a_L^\dagger a_L + 1)$. Then

$$\operatorname{Tr} e^{-tH(E)} = \sum_{n_L, n_R} e^{-tE(2n_L+1)}. \tag{A.12}$$

[12]In this gauge $x^\mu A_\mu = 0$. This will prove useful when we consider the dilation anomaly in Appendix B, since the generator of co-ordinate dilations $x^\mu\partial_\mu$ is gauge invariant: $x^\mu\partial_\mu = x^\mu D_\mu$.

The apparently divergent sum over n_R is just proportional to the area of the (x_1, x_2) space. To see this, suppose that (x_1, x_2) space is confined within a circle of radius L. Then, we should only sum over the harmonic oscillator modes where $\langle n_L, n_R|(x^2 + y^2)|n_L, n_R\rangle = 2/E(n_L + n_R + 1) < L^2$. It is then trivial to perform the sum in the above, and we find

$$\begin{aligned} \mathrm{Tr}\, e^{-tH(E)} &= \frac{L^2 E}{2}\left(\frac{e^{-tE}}{1-e^{-2tE}} + \mathcal{O}\left(\frac{1}{L^2 E}\right)\right) \\ &= \frac{(\pi L^2)E}{4\pi}\frac{1}{\sinh tE} = \frac{1}{4\pi}\int dx_0 dx_1 \frac{E}{\sinh tE}\,, \end{aligned} \tag{A.13}$$

where in the second line we drop all the subleading terms in the large area limit.[13] Finally, then

$$\mathrm{Tr}\, e^{t(D_\mu)^2} = \frac{1}{16\pi^2}\int d^4x EB \frac{1}{\sinh tE \sinh tB}\,. \tag{A.14}$$

The extension to the other heat kernels we need are straightforward. For instance, in the case of the Dirac operator,

$$\not{D}^2 = (D_\mu)^2 - \frac{i}{4}[\gamma^\mu, \gamma^\nu]F_{\mu\nu} = (D_\mu)^2 + \begin{pmatrix} (E+B)\sigma^3 & 0 \\ 0 & (-E+B)\sigma^3 \end{pmatrix} \tag{A.15}$$

and so the heat kernel for left/right handed chiral fermions is

$$\begin{aligned} \mathrm{Tr}_{L,R}\, e^{t\not{D}^2} &= \mathrm{Tr}\, e^{t(D_\mu)^2} \times \mathrm{Tr}\, e^{t(\pm E + B)\sigma^3} \\ &= \frac{1}{16\pi^2}\int d^4x EB \frac{1}{\sinh tE \sinh tB} \times 2\cosh t(E \pm B)\,. \end{aligned} \tag{A.16}$$

Having computed the heat kernels, from eq. (A.7) we obtain,

$$\ln J = \alpha\left(\frac{1}{16\pi^2}\int d^4x EB \frac{2 - 2\cosh t(E+B)}{\sinh tE \sinh tB}\right). \tag{A.17}$$

Similarly the jacobian from $\mathcal{D}\bar{\Phi}$ is

$$\ln \bar{J} = \alpha^*\left(\frac{1}{16\pi^2}\int d^4x EB \frac{2 - 2\cosh t(E-B)}{\sinh tE \sinh tB}\right). \tag{A.18}$$

The result is quite interesting in the following respects. First of all, it is free from ultraviolet divergences $t = M^{-2} \to 0$ because of the cancellation between bosonic and fermionic degrees of freedom, and is well-defined. Expanding the jacobians in the inverse power of cutoff, we have

$$\ln J = \alpha \frac{1}{16\pi^2}\int d^4x \left(-(E+B)^2 + \frac{(E^2 - B^2)^2}{12M^4} + O(M^{-8})\right). \tag{A.19}$$

[13] This result can be obtained with no approximations if we consider the system on a torus.

In supersymmetric notation,

$$\ln J = -\frac{1}{16}\int d^2\theta \frac{2t_2(\Phi)}{8\pi^2}\ln(e^\alpha)W_\alpha W^\alpha + O\left(\frac{1}{M^4}\right), \qquad \text{(A.20)}$$

while the higher order terms can be written as D-terms, $\int d^4\theta (W_\alpha W^\alpha)(\bar{W}_{\dot\beta}\bar{W}^{\dot\beta})/M^4$ etc. In conventional analyses of anomalies, one drops all terms suppressed by powers of cutoff. However, one must keep all higher dimension operators in wilsonian effective actions with a finite ultraviolet cutoff. This implies that the rescaling anomaly is not one-loop exact.[14] We will see later, however, that the "holomorphic" part of J is actually one-loop exact. Note also that the anomalous jacobians are non-trivial even for topologically trivial background gauge fields, e.g., $E \neq 0$ and $B = 0$.

Second, it is useful to check the result with a pure imaginary $\alpha = i\theta$ because it is then a phase change of the chiral superfield Φ and the anomalous jacobian reduces to that of the chiral anomaly. The jacobian is

$$\begin{aligned}\ln J + \ln \bar{J} &= i\theta\left(\frac{1}{16\pi^2}\int d^4x EB\frac{-2(\cosh t(E+B) - \cosh t(E-B))}{\sinh tE \sinh tB}\right)\\ &= i\theta\left(\frac{1}{16\pi^2}\int d^4x(-4)EB\right). \end{aligned} \qquad \text{(A.21)}$$

This is nothing but the second Chern class $F_{\mu\nu}\tilde{F}^{\mu\nu}/16\pi^2 = 4EB/16\pi^2$, and is indeed the correct formula for the chiral anomaly. It is t-independent and does not depend on the precise manner in which the jacobian is regularized. The reason behind the t-independence is its topological nature; the jacobian is actually an integer which corresponds to the mismatch between the number of zero modes for different chiralities. It is believed that the jacobian for the phase rotation is exact for this reason.

Finally, J simplifies drastically under an instanton background, $E = \pm B$. If $E = -B$, the integrand vanishes and there is no anomaly (but $\ln \bar{J} \neq 0$). On the other hand if $E = +B$, the jacobian becomes t-independent,

$$\ln J = \alpha\left(\frac{1}{16\pi^2}\int d^4x EB(-4)\right). \qquad \text{(A.22)}$$

The result under the instanton background can be understood in terms of the zero modes, analogously to the case of the chiral anomaly. First of all, an instanton background preserves half of the supersymmetry. Depending on $E = -B$ or

[14] In wilsonian effective actions, the loop calculations are exhausted at one-loop when one integrates out an infinitesimal slice in the momentum space [10]. However, the one-loop results produce higher dimension operators and they produce corrections to renormalizable operators when one contracts some of the fields in the higher dimension operators. Following the same reasonings, the existence of higher dimension operators in the jacobians suggests that there are higher loop effects. As we have argued in section 2, however, for supersymmetric theories, these higher dimension operators can never feed back into the coefficient of F terms like WW, and so are not relevant to the running of the gauge coupling. They may modify renormalizable D terms such as the kinetic term for the matter fields.

$E = B$, either W^α or $\overline{W}^{\dot\alpha}$ vanishes, and hence either Q^α or $\bar{Q}^{\dot\alpha}$ supercharges are unbroken [7]. Therefore, the modes of the differential operators $(D_\mu)^2$ and $\not{D}^2$ have the same spectrum, and there is a cancellation of eigenvalues between bosonic and fermionic determinants [8]. Let us see this more explicitly. The scalar field can be expanded in terms of the eigenmodes of $(D_\mu)^2$ operator, $-(D_\mu)^2\phi_n = \lambda_n^2\phi_n$. On the other hand, the squared Dirac operator is given in the Weyl basis by

$$(\not{D})^2 = (D_\mu)^2 - \frac{1}{2}\sigma^{\mu\nu}F_{\mu\nu} = \begin{pmatrix} (D_\mu)^2 - \sigma^{\mu\nu}F_{\mu\nu}/2 & 0 \\ 0 & (D_\mu)^2 \end{pmatrix}, \tag{A.23}$$

where we used the fact $\bar{\sigma}^{\mu\nu}F_{\mu\nu} = \bar{\sigma}^{\mu\nu}(F_{\mu\nu} - \tilde{F}_{\mu\nu})/2 = 0$ for an instanton. Therefore, there are two eigenmodes

$$\bar{\psi}_n^1 = \begin{pmatrix} 0 \\ 0 \\ \phi_n \\ 0 \end{pmatrix}, \qquad \bar{\psi}_n^2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \phi_n \end{pmatrix}, \tag{A.24}$$

with the eigenvalue $-(\not{D})^2 = -(D_\mu)^2 = \lambda_n^2$ for this chirality. The eigenmodes of $(\not{D})^2$ with the opposite chirality are given by

$$\psi_n^1 = \frac{1}{\lambda_n} i\not{D}\bar{\psi}_n^1, \qquad \psi_n^2 = \frac{1}{\lambda_n} i\not{D}\bar{\psi}_n^2. \tag{A.25}$$

There are, however, zero modes of $(D_\mu)^2 - \sigma^{\mu\nu}F_{\mu\nu}/2$ which cannot be written in this form because they are not paired with the opposite chirality spinor. We refer to them as ψ_0^i with $i = 1, \ldots, n_0$, where n_0 is the number of zero modes. Finally, F can be expanded in the same eigenmodes of $(D_\mu)^2$ as the scalar component, $F_n = \phi_n$. The path integral measure then reduces to the following form:

$$\mathcal{D}\Phi = \prod_n (d\phi_n d\psi_n^1 d\psi_n^2 dF_n) \prod_i d\psi_0^i. \tag{A.26}$$

Under the rescaling $\Phi = e^\alpha \Phi'$, the jacobians from ϕ_n, $\psi_n^{1,2}$ and F_n precisely cancel: $d\phi_n d\psi_n^1 d\psi_n^2 dF_n = d\phi'_n d\psi_n'^1 d\psi_n'^2 dF'_n$. However the jacobians from the zero modes remain: $\prod_i d\psi_0^i = e^{-n_0\alpha}\prod_i d\psi_0'^i$. This is why the anomalous jacobian is given by the second Chern class; it is the number of zero modes due to the index theorem.[15]

The true anomalous jacobian (that is, the correct change of the bare lagrangian after rescaling which keeps the physics fixed) is in general a complicated function of the field strength W_α and $\bar{W}_{\dot\alpha}$. However, the instanton method shows that the part of

[15]Of course, the Atiyah-Singer index theorem tells us only the difference in the number of zero modes between two chiralities which is a topological invariant. For certain configurations, there may be extra accidental zero modes with equal numbers for both chiralities. In this case, the same zero modes appear also for ϕ and F at the same time, and the accidental zero modes do not contribute to the anomalous jacobian.

the jacobian which is holomorphic in W is *exact* (similarly for anti-holomorphic part). An instanton background has only $W \neq 0$ with $\bar{W} = 0$. Therefore calculation in an instanton background determines the holomorphic part of the anomalous jacobian, and just counts the number of zero modes in the background. The part of the jacobian depending only on W is hence cutoff independent, and is expected to be exact due to the same reasonings as the chiral anomaly case. Indeed, the result cannot be modified by the higher order perturbative corrections because the half of the supersymmetry left unbroken guarantees the cancellation of higher order corrections [8]. Since the jacobian in the instanton background determines the holomorphic dependence on W, we learn that the F-term in the jacobian is exact for arbitrary background.

Even though we have used component calculations, we must mention that the gaussian regularization we used in this subsection can be made manifestly supersymmetric, as was done originally in [13]:

$$\ln J = \alpha \mathrm{STr}\left(e^{tL}\frac{-\bar{D}^2}{4}\right), \tag{A.27}$$

where the factor $-\bar{D}^2/4$ restricts the trace over the superspace only to the chiral one, with[16]

$$L = \frac{1}{16}\bar{D}^2 e^{-2V} D^2 e^{2V}\,. \tag{A.28}$$

Indeed, the operator L reduces to $(D_\mu)^2$ both on the scalar and F-components, while it is $\not{D}^2$ on the spinor component. We can heuristically understand how this L operator can be arrived at in a manifestly supersymmetric way. In a trivial background, we found in the last subsection that an appropriate gaussian cutoff was provided by the operator $\bar{D}^2 D^2/16$ which reduces to $\Box$ on chiral superfields. We are looking for a gauge-covariant extension of this operator. Since we will be taking the trace over the chiral space, our candidate operator L should transform as $L \to e^{i\Lambda} L e^{-i\Lambda}$ under gauge transformation. This is clearly satisfied by $16L = \bar{D}^2 e^{-2V} D^2 e^{2V}$; under the gauge transformation $e^{2V} \to e^{i\bar{\Lambda}} e^{2V} e^{-i\Lambda}$, and

$$\bar{D}^2 e^{-2V} D^2 e^{2V} \to \bar{D}^2 e^{i\Lambda} e^{-2V} e^{-i\bar{\Lambda}} D^2 e^{i\bar{\Lambda}} e^{2V} e^{-i\Lambda} = e^{i\Lambda}\left(\bar{D}^2 e^{-2V} D^2 e^{2V}\right) e^{-i\Lambda}\,. \tag{A.29}$$

We can motivate this operator in another way. In the component analysis, the operator in the gaussian damping was related to operators appearing in the equations of motion, so we can try to get a hint for the form of a manifestly supersymmetric operator by looking at the supersymmetric equations of motion, which are $D^2 e^{2V}\phi = 0$. Of course, we can not use the operator $D^2 e^{2V}$ directly in damping the chiral jacobians, since it maps chiral fields to anti-chiral ones. However, we can get an operator mapping chiral to chiral fields by acting on the left with a $\bar{D}^2$ appropriately;

[16] When we write D^2, it means either the square of the supercovariant derivative, or the square of the D-component in the vector multiplet. We hope they can be easily distinguished according to the context. The gauge covariant derivative is always written as $(D_\mu)^2$.

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as we have seen we need to use $\bar{D}^2 e^{-2V}$ acting on the left to insure gauge covariance of the operator. These heuristic tools for finding manifestly supersymmetric regulators will prove useful in appendix A.5, where we examine the $N = 2$ structure of the anomalies induced in rescaling hypermultiplets in $N = 2$ theories.

One further check which can be made with component calculations is to look at the rescaling jacobian when there is a constant D term background, and compare to the the case with E, B background; they should combine appropriately into $\int d^2\theta WW$. Decomposing the L operator above in the D-term background, we find $L = \Box - D$ on the scalar, $L = \Box$ for on the spinor and $L = \Box + D$ on the F-component of the chiral superfield. The anomalous jacobian under the rescaling is

$$\ln J = -\alpha \left(\mathrm{Tr}_\phi\, e^{-t(-\Box+D)} - \mathrm{Tr}_\psi\, e^{-t\Box} + \mathrm{Tr}_F\, e^{-t(-\Box-D)} \right) = -\alpha \frac{1}{16\pi^2} D^2 \,. \tag{A.30}$$

This contribution is exactly what one expects from $\int d^2\theta WW$ operator with the same normalization as for the case of the E, B background.

A.4 Vector multiplets

In this subsection, we calculate the anomalous jacobian of vector multiplets under rescaling. The basic idea in the calculation is the following. At a given configuration of the gauge field in the functional space, the path integral measure is defined by the top form on the cotangent space, modulo the directions of gauge degrees of freedom. The calculation of the jacobian requires only the knowledge on local properties around each point in the functional space. Therefore, we define the measure in terms of local "fluctuations" around a particular "background" configuration and regularize it in terms of a "background-gauge invariant" operator. Note that we are not employing a background gauge field in the sense of background field formalism where the background is an external classical field. The "background" configuration in the calculation is what needs to be integrated over when the full functional integral is done. However all the steps of the calculation strongly resembles the background field formalism.

As emphasized in appendix A.2, it is important to retain the structure of off-shell multiplet in order to retain the supersymmetric cancellation; in Wess-Zumino gauge we need the gauge field V_μ, gaugino λ and the auxiliary field D. Three (transverse) components of A_μ and D balance the four components of λ, $\bar{\lambda}$ off-shell. As discussed in section 2, we are interested in the anomalous jacobian when one rescales the vector multiplet to bring its kinetic term to the canonical normalization.

We will work with the vector multiplet in the Wess-Zumino gauge, (V_μ, λ, D). The anomalous jacobians from the path integral measures of λ, $\bar{\lambda}$ and D can be readily calculated using the formulae presented in the previous subsection. The discussion of the vector field requires care, since it is only the transverse components which are included in the off-shell multiplet. One can go through supersymmetric

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gauge fixing; it however requires three Faddeev-Popov chiral superfields and many unphysical auxiliary components with higher derivative kinetic terms. We find it more intuitive to work within the Wess-Zumino gauge with appropriate projection on the transverse components. We work on the vector field with the background field formalism, and discuss the jacobian from the path integral measure of the "quantum" vector field V^μ. We will come back to a manifestly supersymmetric method later.

Let us go through the conventional Faddeev-Popov procedure to reduce the path integral volume of the quantum vector field only to its transverse components, being careful to keep track of the normalization of the path integral. As usual, we insert an identity

$$1 = \int \mathcal{D}g\, \delta(D^\mu V^g_\mu - a) \det(D^\mu D_\mu) \tag{A.31}$$

into the path integral, and rewrite the determinant factor using the Faddeev-Popov ghost. Here, V^g is a gauge transformed vector field according to the gauge function $g(x)$, and $D_\mu = \partial_\mu - iA_\mu$ with respect to the background vector field A_μ. The gauge group volume $\mathcal{D}g$ can be normalized to unity, and it can be dropped from the path integral because the rest of the integrand is gauge-invariant. Care must be taken when "smearing" the gauge fixing condition $D^\mu V_\mu = a$ over the arbitrary space-time dependent function $a(x)$. To obtain the desired gauge fixing term $\frac{\xi}{2g^2}(D^\mu V_\mu)^2$, one must integrate over a as

$$e^{-\int d^4x \frac{\xi}{2g^2}(D^\mu V_\mu)^2} = \frac{1}{N}\int \mathcal{D}a\, \delta(D^\mu V^g_\mu - a) e^{-\int d^4x \frac{\xi}{2g^2}a^2}\,, \tag{A.32}$$

where the normalization factor N depends on the gauge coupling constant, $N = (\det(g^2/\xi))^{-1/2}$. When one rescales the gauge field from holomorphic to canonical normalization, this factor N also changes. To keep track of this factor N, we write the path integral over the gauge field V^μ as

$$\frac{\int \mathcal{D}V\mathcal{D}c\mathcal{D}\bar{c}\, e^{-S_V - S_{FP} - S_{gf}}}{\int \mathcal{D}a\, e^{-\int d^4x \frac{\xi}{2g^2}a^2}}\,, \tag{A.33}$$

where S_V is the action for the quantum gauge field V_μ in the presence of a background, $S_{FP} = \int d^4x \bar{c} D_\mu D^\mu c$ is the Faddeev-Popov term, and $S_{gf} = \int d^4x \frac{\xi}{2g^2}(D_\mu V^\mu)^2$ is the gauge fixing term. Note that the Faddeev-Popov action does not have an overall $1/g^2$ and hence does not need to be rescaled. On the other hand, both the vector field V^μ and the "smearing" factor a need to be rescaled. The kinetic operator for V_μ is given by

$$\begin{aligned}(D_\mu)^2\delta_{\mu\nu} - D_\nu D_\mu - i\frac{1}{2}F_{\rho\sigma}(M^{\rho\sigma})_{\mu\nu} + \xi D_\mu D_\nu &= \\ = (D_\mu)^2\delta_{\mu\nu} - iF_{\rho\sigma}(M^{\rho\sigma})_{\mu\nu} + (\xi - 1)D_\mu D_\nu\,,&\end{aligned} \tag{A.34}$$

where $M^{\rho\sigma}$ are SO(4) rotation generators. The anomalous jacobian from rescaling $V^\mu = e^\alpha V'^\mu$ is given by

$$\begin{aligned}\ln J_V &= \alpha \,\mathrm{Tr}_V\, e^{-t((D_\mu)^2\delta_{\mu\nu} - iF_{\rho\sigma}(M^{\rho\sigma})_{\mu\nu} + (\xi-1)D_\mu D_\nu)} \\ &= \alpha\left(\mathrm{Tr}_{V_T}\, e^{-t((D_\mu)^2\delta_{\mu\nu} - iF_{\rho\sigma}(M^{\rho\sigma})_{\mu\nu})} + \mathrm{Tr}_{V_L}\, e^{-t\xi D_\mu D^\mu}\right), \end{aligned} \tag{A.35}$$

where we have decomposed the space into the transverse one $V_T^\mu = V^\mu - \frac{D^\mu D^\nu}{(D_\rho)^2}V_\nu$ and the longitudinal $V_L^\mu = \frac{D^\mu D^\nu}{(D_\rho)^2}V_\nu$. The jacobian from $\mathcal{D}a$ is regularized uniquely as

$$\ln J_a = \alpha\, \mathrm{Tr}_a\, e^{-t\xi D_\mu D^\mu} \tag{A.36}$$

to guarantee the cancellation of the jacobian under a trivial background, and the combination $\ln J_V - \ln J_a$ is independent of the gauge parameter ξ. Therefore, we can simplify the calculation by taking $\xi = 1$ (Feynman gauge).

The jacobian of the D-component is regularized as

$$\ln J_D = \alpha\, \mathrm{Tr}_D\, e^{t(D_\mu D^\mu)}, \tag{A.37}$$

which is the only one allowed by the gauge invariance. Putting all factors together, the total jacobian is given by

$$\begin{aligned}\ln J = \alpha\Big(&\mathrm{Tr}_V\, e^{t((D_\mu)^2\delta_{\mu\nu} - iF_{\rho\sigma}(M^{\rho\sigma})_{\mu\nu})} - \\ &- \mathrm{Tr}_a\, e^{t(D_\mu)^2} - \mathrm{Tr}_\lambda\, e^{t\not{D}^2} - \mathrm{Tr}_{\bar\lambda}\, e^{t\not{D}^2} + \mathrm{Tr}_D\, e^{t(D_\mu D^\mu)}\Big)\end{aligned} \tag{A.38}$$

and the second term is the same as the last term. We finally find that the anomalous jacobian is simply

$$\ln J = \alpha\left(\mathrm{Tr}_V\, e^{t((D_\mu)^2\delta_{\mu\nu} - iF_{\rho\sigma}(M^{\rho\sigma})_{\mu\nu})} - \mathrm{Tr}_\lambda\, e^{t\not{D}^2} - \mathrm{Tr}_{\bar\lambda}\, e^{t\not{D}^2}\right). \tag{A.39}$$

To simplify the analysis, we take SU(2) gauge group, and take a constant background field strength in W^3 gauge field. W^+ carries a positive charge unity under the background. The rest is the calculation of the jacobian from W^+ multiplet only. The only new heat kernel we need is

$$\begin{aligned}\mathrm{Tr}_V\, e^{t((D_\mu)^2\delta_{\mu\nu} - iF_{\rho\sigma}(M^{\rho\sigma})_{\mu\nu})} &= \mathrm{Tr}\, e^{t(D_\mu)^2} \times \mathrm{Tr}\,\exp\begin{pmatrix} 2itE\sigma^2 & 0 \\ 0 & 2itB\sigma^2\end{pmatrix} \\ &= \frac{1}{16\pi^2}\int d^4x EB \frac{2(\cosh 2tE + \cosh 2tB)}{\sinh tE \sinh tB}. \end{aligned} \tag{A.40}$$

The anomalous jacobian of the vector multiplet $\mathcal{D}V = \mathcal{D}(e^\alpha V') = \mathcal{D}V' J$ is given by

$$\ln J = -\alpha \frac{1}{16\pi^2}\int d^4x EB \frac{2(\cosh 2tE + \cosh 2tB) - 4\cosh tE \cosh tB}{\sinh tE \sinh tB}. \tag{A.41}$$

As expected, there is no ultraviolet divergence $t = M^{-2} \to 0$. By expanding the expression in powers of t, one finds

$$\ln J = \alpha \frac{1}{16\pi^2} \int d^4x \left(2(E^2 + B^2) + \frac{5}{6} \frac{(E^2 - B^2)^2}{M^4} + O(M^{-8}) \right) \tag{A.42}$$

The finite part is exactly opposite to the contribution of a chiral superfield with the same charge.

As in the case of chiral multiplets, the jacobian simplifies drastically for an instanton background $E = B$, where it becomes

$$\ln J = -\alpha \frac{1}{16\pi^2} \int d^4x EB \times 4 \tag{A.43}$$

and is cutoff independent. This is again a consequence of the zero modes. We have discussed the zero modes of spinors already in the previous subsection. The eigenvalues of the operator $(D_\mu)^2 + iM_{\rho\sigma}F^{\rho\sigma}$ are the same as the squared Dirac operator except the zero modes. It is useful to write a vector field V_μ as a bi-spinor $V_{\alpha\dot{\alpha}}$ for this purpose. The eigenmodes satisfy the equation

$$-(D_\mu)^2 V + \sigma_{\rho\sigma}F^{\rho\sigma}V + V\bar{\sigma}^T_{\rho\sigma}F^{\rho\sigma} = \lambda_n^2 V \,. \tag{A.44}$$

The point is that $\bar{\sigma}_{\rho\sigma}F^{\rho\sigma} = \bar{\sigma}_{\rho\sigma}(F^{\rho\sigma} - \tilde{F}^{\rho\sigma})/2 = 0$ for the instanton background. Therefore, the eigenequation becomes exactly the same as that of spinors except a left-over free spinor index $\dot{\alpha}$. The eigenvalues of the vector multiplet are exactly the same as those of the spinor λ with twice as much degeneracy. Together with the analysis in the previous subsection, we find the following spectrum. For each non-zero eigenvalue of $-(D_\mu)^2 = \lambda_n^2$, there are two modes for λ, two modes for $\bar{\lambda}$, four modes for V_μ. However one of the four modes for V_μ is longitudinal. Since the longitudinal mode is always accompanied by the corresponding mode in $\mathcal{D}a$ and the jacobians cancel between them, we drop it from the discussion. For n_0 zero modes for λ, there are $2n_0$ zero modes for V_μ, and all of them are transverse. Therefore in the instanton background, the path integral measure reduces to the following:

$$\int \mathcal{D}V = \int \prod_n \left(dV_n^1 dV_n^2 dV_n^3 d\lambda_n^1 d\lambda_n^2 d\bar{\lambda}_n^1 d\bar{\lambda}_n^2 dD_n \right) \prod_i^{2n_0} dV_0^i \prod_i^{n_0} d\lambda_0^i \,. \tag{A.45}$$

When one rescales the whole vector multiplet, the contributions from all non-zero modes cancel among themselves. The anomaly under the rescaling is therefore determined by $2n_0 - n_0$, which is the opposite of the case of a chiral superfield. Following the same reasoning as in the previous subsection, the part of the jacobian holomorphic in W is exact.

The final result of the jacobian for a general non-abelian gauge group is

$$\mathcal{D}(g_c V_c) = \mathcal{D}(V_c) \exp\left(\frac{1}{16} \int d^4y \int d^2\theta \frac{2t_2(A)}{8\pi^2} \ln g_c W^a(g_c V_c) W^a(g_c V_c) + \right.$$
$$\left. + \text{h.c.} + \mathcal{O}\left(\frac{1}{M^4}\right)\right) . \tag{A.46}$$

A manifestly supersymmetric formulation of the jacobian is possible in the background field formalism [38] but is technically more complicated. First of all, one needs three Faddeev-Popov ghosts c, c', and b, which are all chiral superfields. The first two appear in a rather conventional manner. The delta functional for gauge fixing $\delta(\bar{\mathcal{D}}^2 V - a)\delta(\mathcal{D}^2 V - \bar{a})$ cannot be inserted to the path integral by itself because it varies along the gauge volume. Here and below, $\mathcal{D}_\alpha = e^{2W^B} \bar{D}_\alpha e^{-2W^B}$ is the background-chiral supercovariant derivative, with $e^{2W^B} e^{2\bar{W}^B} = e^{2V^B}$ is the background vector multiplet. The gauge variation of the gauge field V is given by $e^{2V'} = e^{i\bar{\Lambda}} e^{2V} e^{-i\Lambda}$, or $2\delta V = -iL_V[(\bar{\Lambda} + \Lambda) + \coth(L_V)(\Lambda - \bar{\Lambda})]$ for infinitesimal Λ, $\bar{\Lambda}$. (L_V acts as $L_V c = [V, c]$ etc, and the formal expression $L_V \coth L_V$ is understood in terms of its Taylor expansion.) Therefore, one inserts the combination

$$\delta(\bar{\mathcal{D}}^2 V - a)\delta(\mathcal{D}^2 V - \bar{a}) \int \mathcal{D}c\mathcal{D}\bar{c}\mathcal{D}c'\mathcal{D}\bar{c}' e^{\mathrm{Tr}\int d^4x d^4\theta (\bar{c}' - c') L_V[(\bar{c}+c) + \coth(L_V)(c - \bar{c})]} . \quad \text{(A.47)}$$

Note that the ghost fields c, $\bar{c}$ have the normalization of the gauge parameters and hence do not have $1/g^2$ in front of the lagrangian. The third one b corresponds to the normalization factor from a-integration in the component treatment. It appears when one "smears" over the gauge fixing condition $\bar{\mathcal{D}}^2 V = a$ where a is a chiral superfield. To guarantee that the delta functional is correctly replaced by a path integral without any additional factors, one needs to compensate the integral over a by an integral over ghost b,

$$\begin{aligned} \delta(\bar{\mathcal{D}}^2 V - a)\delta(\mathcal{D}^2 V - \bar{a}) &\to \int \mathcal{D}a\mathcal{D}\bar{a}\mathcal{D}b\mathcal{D}\bar{b}\delta(\bar{\mathcal{D}}^2 V - a)\delta(\mathcal{D}^2 V - \bar{a}) e^{-\frac{1}{16g^2}\int d^4x d^4\theta (\bar{a}a + \bar{b}b)} \\ &= \int \mathcal{D}b\mathcal{D}\bar{b} e^{-\frac{1}{16g^2}\int d^4x d^4\theta ((\mathcal{D}^2 V)(\bar{\mathcal{D}}^2 V) + \bar{b}b)} . \end{aligned} \quad \text{(A.48)}$$

One needs the normalization $1/g^2$ so that the gauge fixing term after the a integral combines with the gauge kinetic term in the holomorphic normalization. The b integral can not be dropped since b is background-chiral, i.e. it satisfies the chirality condition $\bar{\mathcal{D}}_\alpha b = e^{2W^B} \bar{D}_\alpha e^{-2W^B} b = 0$,[17] and hence the path integral over $b, \bar{b}$ depends on the background gauge field.

The change from the holomorphic normalization to the canonical normalization requires rescaling of the full vector multiplet and b-ghost, but not c, c' ghosts. The jacobian from the b-ghost is the same as that from a chiral superfield in the adjoint representation except with opposite sign. The vector multiplet produces jacobians from all components, i.e., $V = C + i\theta\chi - i\bar{\theta}\bar{\chi} + i\theta^2(M + iN)/2 - i\bar{\theta}^2(M - iN)/2 - \theta\sigma^\mu\bar{\theta}V_\mu + i\theta^2\bar{\theta}(\bar{\lambda} + i\bar{\partial\!\!\!/}\chi/2) - i\bar{\theta}^2\theta(\lambda + i\partial\!\!\!/\bar{\chi}/2) + \theta^2\bar{\theta}^2(D/2 + \Box C/4)$. The jacobian is regularized by the kinetic operator,

$$\ln J = \alpha \mathrm{STr}_V e^{t((D_\mu)^2 - \mathcal{W}^\alpha \mathcal{D}_\alpha + \bar{\mathcal{W}}^{\dot{\alpha}} \bar{\mathcal{D}}_{\dot{\alpha}})} , \quad \text{(A.49)}$$

[17] If necessary, one can rescale b-ghost to absorb the factor of 16 by properly changing the holomorphic gauge coupling constant $8\pi^2/g_h^2$ by $-C_A \ln 16$.

which reduces to $(D_\mu)^2$ for C, M, N, D components, $(D_\mu)^2 + M_{\rho\sigma}F^{\rho\sigma}$ for the vector V_μ, and $\not{D}^2$ for χ, λ. Therefore, the addition to the case in the Wess-Zumino gauge is C, M, N, V_L (longitudinal component of V_μ) and χ, $\bar{\chi}$, and hence is the same as an extra chiral superfield in the adjoint representation. This additional contribution is precisely canceled by the jacobian from the b-ghost and hence our component calculation is justified from a manifestly supersymmetric framework. Put another way, the vector multiplet does not produce an anomalous jacobian in a manifestly supersymmetric analysis; this is because one needs two powers of $\mathcal{D}_\alpha$ and two powers of $\bar{\mathcal{D}}_{\dot{\alpha}}$ to get a non-vanishing supertrace, and the leading term is hence $WW\bar{W}\bar{W}$ which is a higher-dimensional D-term. The relevant jacobian comes solely from the b-ghost; therefore it is always the opposite of that from a chiral superfield in the adjoint representation.

A.5 $N = 2$ invariance

In the previous subsections, we calculated anomalous jacobians of the chiral and vector multiplet in $N = 1$ supersymmetric gauge theories. A natural question is what happens when one studies theories with extended supersymmetries. Clearly, the gaussian cutoff method can be extended for the rescaling anomaly of hypermultiplets in $N = 2$ theories. An important question then is whether the rescaling of a hypermultiplet produces both the $\int d^2\theta WW$ operator and the kinetic term of the adjoint superfield $\int d^4\theta 2\,\mathrm{Tr}\,\bar{\Phi}e^{2V}\Phi e^{-2V}$ needed to preserve $N = 2$ supersymmetry. Of course, the hypermultiplets do not receive wave-function renormalization and its rescaling is not necessary for the computation of β-function. However, the rescaling of a hypermultiplet in the adjoint representation is necessary to derive the Shifman-Vainshtein formula eq. (2.10) from the $N = 1$ SUSY YM regularized by $N = 4$ theory in section 6, and it is important to check that the jacobian preserves extended supersymmetry. The $N = 4$ invariance of the jacobian needed in section 6 follows trivially once the $N = 2$ invariance is verified.

There is a superpotential coupling of a hypermultiplet $(Q, \tilde{Q})$ to the adjoint superfield Φ in the vector multiplet,

$$\int d^2\theta \sqrt{2}\tilde{Q}\Phi Q\,. \tag{A.50}$$

To see that $\int d^4\theta 2\,\mathrm{Tr}\,\bar{\Phi}e^{2V}\Phi e^{-2V}$ is generated from rescaling the hypermultiplet, we need to employ a background configuration of Φ such that the kinetic operator does not vanish. A convenient choice is when the F-component of Φ does not vanish. For simplicity, we discuss $N = 2$ supersymmetric QED, where Φ has only one component and is electrically neutral.

Let us first find a manifestly $N = 2$ supersymmetric gaussian damping operator. In order to do this, we follow the strategy used in appendix A.3 and use the supersymmetric equations of motion to infer the form of the operator we need, this should work since the equations of motion are certainly $N=2$ covariant. The equations of motion are

$$\begin{pmatrix} D^2 e^{2V} & \sqrt{2}\bar{\Phi} \\ \sqrt{2}\Phi & \bar{D}^2 e^{-2V} \end{pmatrix} \begin{pmatrix} Q \\ \widetilde{Q}^\dagger \end{pmatrix}. \tag{A.51}$$

As before, in order to find an operator that correctly maps (anti) chiral to (anti) chiral fields, and which is moreover gauge covariant, we form

$$\begin{aligned} L \begin{pmatrix} Q \\ \widetilde{Q}^\dagger \end{pmatrix} &= \frac{1}{16} \begin{pmatrix} \bar{D}^2 e^{-2V} & 0 \\ 0 & D^2 e^{2V} \end{pmatrix} \begin{pmatrix} D^2 e^{2V} & \sqrt{2}\bar{\Phi} \\ \sqrt{2}\Phi & \bar{D}^2 e^{-2V} \end{pmatrix} \begin{pmatrix} Q \\ \widetilde{Q}^\dagger \end{pmatrix} \\ &= \frac{1}{16} \begin{pmatrix} \bar{D}^2 e^{-2V} D^2 e^{2V} & \sqrt{2}\bar{D}^2 e^{-V}\bar{\Phi} \\ \sqrt{2} D^2 e^{2V}\Phi & D^2 e^{2V}\bar{D}^2 e^{-2V} \end{pmatrix} \begin{pmatrix} Q \\ \widetilde{Q}^\dagger \end{pmatrix}. \end{aligned} \tag{A.52}$$

For trivial gauge fields and a background F component F_Φ for Φ, the action of L on components is very simple: $\Box$ on the fermion and F components of $(Q, \tilde{Q})$, and

$$L \begin{pmatrix} A_Q \\ \bar{A}_{\widetilde{Q}} \end{pmatrix} = \begin{pmatrix} \Box & \sqrt{2}\bar{F}_\Phi \\ \sqrt{2}F_\Phi & \Box \end{pmatrix} \begin{pmatrix} A_Q \\ \bar{A}_{\widetilde{Q}} \end{pmatrix} \tag{A.53}$$

on the A components of $(Q, \widetilde{Q})$, so L has eigenvalues $\Box \pm \sqrt{2F_\Phi \bar{F}_\Phi}$ on the space of A components. The jacobian is then

$$\begin{aligned} \ln J &= \alpha \,\mathrm{Tr}\left(e^{t(\Box + \sqrt{2F_\Phi\bar{F}_\Phi})} + e^{t(\Box - \sqrt{2F_\Phi\bar{F}_\Phi})} - 2e^{t\Box} \right) \\ &= \alpha \int d^4x \frac{d^4p}{(2\pi)^4} \left(e^{-t(p^2 + \sqrt{2F_\Phi\bar{F}_\Phi})} + e^{-t(p^2 - \sqrt{2F_\Phi\bar{F}_\Phi})} - 2e^{-tp^2} \right) \\ &= \alpha \frac{1}{4\pi^2} \int d^4x F_\Phi \bar{F}_\Phi + O\left(\frac{1}{M^4}\right). \end{aligned} \tag{A.54}$$

This is nothing but the kinetic term of the adjoint superfield $-\bar{F}_\Phi F_\Phi$ multiplied by $-\alpha/4\pi^2$. On the other hand the corresponding jacobian in a gauge field background is

$$\begin{aligned} \ln J &= \alpha \frac{-1}{16\pi^2} \int d^4x \left((E+B)^2 + (E-B)^2 \right) + O\left(\frac{1}{M^4}\right) \\ &= \alpha \frac{-1}{4\pi^2} \int d^4x \left(\frac{1}{2}(E^2 + B^2) \right) + O\left(\frac{1}{M^4}\right), \end{aligned} \tag{A.55}$$

which is again the gauge kinetic term multiplied by $-\alpha/4\pi^2$. Therefore the anomalous jacobian which we calculated comes out $N=2$ supersymmetric automatically.

B. Trace anomalies

In this appendix, we employ the same formalism as in the previous appendix to work out the trace anomaly, or the anomalous jacobians under dilation. The dilation is nothing but the change in the overall mass scale:

$$\phi(x) \to \phi'(x) = e^{d\lambda}\phi(e^{-\lambda}x)\,, \tag{B.1}$$

where d is the canonical dimension of the field: $d = 1$ for Klein-Gordon or vector fields and $d = 3/2$ for spinor fields. The corresponding current is

$$j^\mu_D = x_\nu \theta^{\mu\nu}\,, \tag{B.2}$$

where $\theta^{\mu\nu}$ is the symmetric (improved) energy momentum tensor [40]. The classical lagrangians with no dimensionful parameters have invariance under the dilation, while quantum mechanically the presence of the cutoff destroys the scale invariance, and there is a trace anomaly,

$$\partial_\mu j^\mu_D = \theta^\mu_\mu \neq 0\,. \tag{B.3}$$

The infinitesimal dilation can be written as

$$\delta\phi(x) = \lambda(d - x^\mu \partial_\mu)\phi(x)\,. \tag{B.4}$$

For scalar fields, the regularized jacobian for an infinitesimal dilation is then given by[18]

$$\ln J = \lambda \operatorname{Tr}\left((d - x_\mu \partial^\mu_x)e^{t(D_\mu)^2}\right) = \operatorname{Tr}\left((d - 2 - \frac{1}{2}\{x_\mu, \partial^\mu_x\})e^{t(D_\mu)^2}\right), \tag{B.5}$$

where we have used $-x_\mu \partial^\mu_x = -1/2[x_\mu, \partial^\mu_x] - 1/2\{x_\mu, \partial^\mu_x\} = -2 - 1/2\{x_\mu, \partial^\mu_x\}$. Note that, as remarked in the previous appendix, in the case with constant background electric and magnetic fields, we found a gauge with $x^\mu A_\mu = 0$, so that in fact the operator $x^\mu \partial_\mu = x^\mu D_\mu$ appearing in the jacobian is gauge covariant.

It is easy to see that the anti-commutator piece does not contribute to the trace in the jacobian. Since the eigenstates of $(D_\mu)^2$ (in the constant E, B background we are considering) are harmonic oscillator modes, it suffices to note that $\{x_\mu, \partial^\mu_x\} \sim i(a^2_\mu - a^{\dagger 2}_\mu)$, and so $\langle n_L, n_R|\{x_\mu, \partial^\mu_x\}|n_L, n_R\rangle = 0$ for the $|n_L, n_R\rangle$ harmonic oscillator eigenstates. Therefore,

$$\ln J = (d-2) \operatorname{Tr} e^{t(D_\mu)^2}\,. \tag{B.6}$$

In other words, the anomalous jacobian under a dilation for individual component is exactly the same as under a rescaling, with weight $(d - 2)$. This can shown to be true for the spinor and vector fields as well.

The anomalous jacobian under the dilation can be now easily worked out for a chiral multiplet in a gauge-field background. It is given by

$$\ln J = \lambda\left(-\operatorname{Tr}_\phi e^{t(D_\mu)^2} + \frac{1}{2}\operatorname{Tr}_\psi e^{-t\not{D}^2}\right), \tag{B.7}$$

[18] Traditionally, the anomalous jacobians under dilation were discussed in terms of Weyl transformations [37]. We do not use this method here to avoid going into supergravity extension of the Weyl transformations.

since the auxiliary component F has a canonical dimension $d = 2$, and hence has a vanishing weight $d - 2 = 0$. Using the formulae given in the previous appendix,

$$\ln J = \lambda \frac{-1}{16\pi^2} \int d^4x EB \frac{1 - \cosh t(E \pm B)}{\sinh tE \sinh tB} = \lambda \frac{1}{16\pi^2} \int d^4x \frac{1}{2}(E \pm B)^2 + O\left(\frac{1}{M^4}\right). \tag{B.8}$$

In supersymmetric notation for a general chiral multiplet, it is

$$\ln J = \lambda \frac{1}{16} \int d^4x d^2\theta \frac{t_2(i)}{8\pi^2} WW + O\left(\frac{1}{M^4}\right). \tag{B.9}$$

This jacobian gives the correct one-loop contribution to the holomorphic β-function from the chiral multiplet.

In this analysis, the holomorphy between $\mathrm{U}(1)_R$ transformation and dilation is manifest. The $\mathrm{U}(1)_R$ transformation with charge $2/3$ for the chiral superfield Φ rotates the phases of component fields with charges $2/3$ for ϕ, $-1/3$ for ψ and $-4/3$ for F. The jacobian is therefore

$$\begin{aligned}\ln J &= i\alpha \left(\frac{2}{3} \operatorname{Tr}_\phi e^{t(D_\mu)^2} - \frac{1}{3} \operatorname{Tr}_\psi e^{-t\not{D}^2} - \frac{4}{3} \operatorname{Tr}_F e^{t(D_\mu)^2} \right) \\ &= i\alpha \left(-\frac{2}{3} \operatorname{Tr}_\phi e^{t(D_\mu)^2} - \frac{1}{3} \operatorname{Tr}_\psi e^{-t\not{D}^2} \right),\end{aligned} \tag{B.10}$$

where we used the equality of the traces on ϕ and F components. This is precisely the same as the jacobian under the dilation except a factor of $i2/3$ and $\lambda \to \alpha$. Note that the form above is not t-independent, but the combination $\ln J + \ln \bar{J}$ is, and hence the $\mathrm{U}(1)_R$ anomaly is exact. The F-terms in the jacobians are exact individually for J and $\bar{J}$, as can be seen by employing the instanton background $E = B$,

$$EB \frac{1 - \cosh t(E \pm B)}{\sinh tE \sinh tB}\bigg|_{E=B} = -\frac{1}{2}(E \pm B)^2\,, \tag{B.11}$$

with no t-dependence and given only by the zero modes.

One can go through the same calculation for a vector multiplet around each point in the functional space. In Wess-Zumino gauge, the contributions come from the all four components of V_μ after gauge fixing with weight -1, Faddeev-Popov ghosts c and $\bar{c}$ with weights -1 but with the opposite sign, and gauginos λ and $\bar{\lambda}$ with $-1/2$ but with the opposite sign. Note that auxiliary fields have vanishing weights and hence do not contribute. We find

$$\begin{aligned}\ln J &= \lambda \Big(- \operatorname{Tr}_V e^{t((D_\mu)^2 \delta_{\mu\nu} - iF_{\rho\sigma}(M^{\rho\sigma})_{\mu\nu})} + \operatorname{Tr}_c e^{t(D_\mu)^2} + \\ &\qquad + \operatorname{Tr}_{\bar{c}} e^{t(D_\mu)^2} + \frac{1}{2} \operatorname{Tr}_\lambda e^{-t\not{D}^2} + \frac{1}{2} \operatorname{Tr}_{\bar{\lambda}} e^{-t\not{D}^2} \Big) \\ &= \lambda \frac{-1}{16\pi^2} \int d^4x EB \frac{2(\cosh 2tE + \cosh 2tB) - 2 - \cosh t(E+B) - \cosh t(E-B)}{\sinh tE \sinh tB} \\ &= \lambda \frac{-1}{16\pi^2} \int d^4x 3(E^2 + B^2) + O\left(\frac{1}{M^4}\right).\end{aligned} \tag{B.12}$$

Again the instanton background gives a t-independent result

$$EB\frac{2(\cosh 2tE+\cosh 2tB)-2-\cosh t(E+B)-\cosh t(E-B)}{\sinh tE\sinh tB}=6EB\,, \quad \text{(B.13)}$$

and hence the jacobian is exact for F-terms but not for higher-dimensional D-terms.

It is interesting to compare the above jacobian with that under $\mathrm{U}(1)_R$ transformation. They agree only up to higher dimension D-terms. The jacobian under the $\mathrm{U}(1)_R$ current is simply that from λ with charge $+1$ and $\bar{\lambda}$ with charge -1, and hence

$$\begin{aligned}\ln J &= i\alpha\left(\mathrm{Tr}_\psi\, e^{-t\not{D}^2}-\mathrm{Tr}_{\bar\psi}\, e^{-t\not{D}^2}\right) \qquad \text{(B.14)}\\ &= i\alpha\int d^4x EB\frac{2(\cosh t(E+B)-2\cosh t(E-B))}{\sinh tE\sinh tB}=i\alpha\int d^4x 4EB.\end{aligned}$$

This is again $i2/3$ times that of the trace anomaly. One little surprise here is that the relation between the trace anomaly and $\mathrm{U}(1)_R$ anomaly is not exact, but appears to hold only for finite pieces. This is not a true statement because we do not use a manifestly holomorphic formalism for the vector multiplet. Recall that the total $\mathrm{U}(1)_R$ anomaly for a chiral multiplet had a cancellation of higher-dimension operators between J and $\bar{J}$. Even though the chiral jacobian J preserves manifest holomorphy between $\mathrm{U}(1)_R$ and trace anomalies, the total jacobian $J\bar{J}$ does not. The apparent mismatch between the $\mathrm{U}(1)_R$ and trace anomalies in the explicit forms of the jacobians is an artifact of the formalism. In the $N=4$ regularization of pure SUSY YM we presented in section 5, the holomorphy between $\mathrm{U}(1)_R$ and trace anomalies was manifest.

A manifestly supersymmetric formalism requires three sets of ghost chiral superfields, b, c and c' as reviewed in appendix A.4. The contribution from the ghost fields is the same as for a chiral multiplet with an overall multiplicative factor -3. In this formalism the holomorphy between $\mathrm{U}(1)_R$ anomaly and trace anomaly is manifest as well for the contributions from the ghost chiral superfields. There is no contribution from the full vector multiplet to the $\mathrm{U}(1)_R$ anomaly. There is, however, a contribution from the full vector multiplet to the trace anomaly with the following weights: $C(-2)$, $\psi(-3/2)$, $\bar{\psi}(-3/2)$, $V_\mu(-1)$, $M(-1)$, $N(-1)$, $\lambda(-1/2)$, $\bar{\lambda}(-1/2)$ and $D(0)$. One finds

$$\begin{aligned}\ln J_V &= \lambda\frac{-1}{16\pi^2}\int d^4x EB\frac{4+2(\cosh 2tE+\cosh 2tB)-2(\cosh t(E-B)+\cosh t(E+B))}{\sinh tE\sinh tB}\\ &= \lambda\frac{-1}{16\pi^2}\int d^4x\frac{(E^2-B^2)^2}{M^4}+O\left(\frac{1}{M^8}\right). \qquad \text{(B.15)}\end{aligned}$$

Therefore, the leading contribution is a higher dimensional D-term $\int d^4\theta\frac{(WW)(\bar{W}\bar{W})}{M^4}$ where M is the ultraviolet cutoff, which can be dropped when one studies the running gauge coupling constant. The above combination trivially vanishes under an instanton background $E=\pm B$.

The final answer for the jacobian of the vector multiplet for an arbitrary gauge group is given in supersymmetric notation by

$$\ln J = \lambda \frac{1}{16} \int d^4x d^2\theta \frac{-3t_i(A)}{8\pi^2} WW + \text{h.c.} + O\left(\frac{1}{M^4}\right). \tag{B.16}$$

This jacobian gives the correct one-loop contribution to the holomorphic β-function from the vector multiplet.

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